STRATEGIC INFORMED TRADING AND THE VALUE OF PRIVATE INFORMATION

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Abstract. We consider a market of risky financial assets consisting of an informed trader, a mass of uniformed traders and noisy liquidity providers. We prove the existence of market-clearing equilibrium when the insider internalizes the power to impact prices. At this equilibrium, the insider strategically reveals a noisier signal, making the prices less reactive to the publicly available information. In contrast to the related literature, we show that within price-impact equilibrium, the insider's ex-ante welfare is monotonically increasing in the signal precision. This clarifies at which situations a trader with market power has motive to obtain private information of good quality. Even though uniformed traders act as price-takers, the effect of price impact is beneficial for them as long as they are at the same side of trading with the insider. On the other hand, in the presence of asymmetric information, price impact equilibrium may yield lower insider's welfare if she is sufficiently risk averse, while uniformed traders are sufficiently risk tolerant.

Introduction

It is well-documented fact that large financial institutions do possess the power to affect markets due to their sizes (see e.g. the empirical evidence in Kacperczyk and Pagnotta [2019] and the related discussion in Rostek and Yoon [2023]). Compared to the rest of the traders, large investors' orders impact transaction prices and volumes, meaning that we can not consider them as price-takers (Rostek and Weretka [2015a]). In addition to their market power, they normally invest considerable capital on acquisition of better information with regards to the payoffs of the assets they trade (Kacperczyk and Pagnotta [2019]). In other words, besides their price impact, large investors can also be seen as "insiders", in the sense that they possess asymmetric information compared to the rest of the traders.

On the other hand, the rest of the market knows that the large investors are informed traders (as in Subrahmanyam [1991]). This means that their demand takes into the fact that the insiders have better information about the tradeable assets. Indeed, under the presence of noise traders, an insider's private signal is partially revealed to the uniformed traders through the equilibrium prices, a process that creates the so-called market (or public) signal (as in e.g. Kyle [1985]). Based on the above, it is reasonable to assume that an insider who is also endowed with market power will strategically choose the signal that she reveals to the market aiming to increase her own gains from the transaction, while uniformed traders recognise the fact that insider internalize her price impact.

Our main goal is to study the effect of price impact induced by an insider's maker power to the equilibrium prices, information transmission and traders' welfare. For this, we adapt to the classic normal-CARA setting a linear price-impact equilibrium where a risk averse insider endowed with private information trades a bundle of risky assets with a mass of uniformed risk-averse traders and noisy liquidity providers (noise traders). We provide model's predictions regarding insider's motives to wangle her revealed signal, the way the uniformed traders adjust their demands and ask whether the informativeness of equilibrium prices is reduced due to insider's strategy. Furthermore, a welfare analysis indicates when price impact increases traders' welfare and whether private signal of a better quality leads to insider's higher gains from trading and hence to higher valuation of private information.

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Main Contributions. The assumption that insiders take prices as given is ubiquitous in the literature on equilibrium under heterogeneous information. Indeed, this assumption was made in the seminal papers of Grossman [1976], Grossman and Stiglitz [1980] and Hellwig [1980], and with the exception of the literature strand started by Kyle [1985], Back [1992] Rochet and Vila [1994] and Subrahmanyam [1991], price taking has remained the dominant assumption.

In the setting of Grossman and Stiglitz [1980], we consider an equilibrium, where an insider internalizes her price impact assuming the presence of price-taker uniformed traders and noise traders.[1] Following the related literature, we study linear impact equilibrium in the sense that the equilibrium price vector is an affine function of insider and noise traders' demand[2], with its coefficients being endogenously derived through market clearing.

We firstly provide sufficient conditions for the existence of the price-impact equilibrium, which are always valid in the univariate case (Theorem 2.8). The affine coefficients are not the same as in the price-taking case, and in fact are governed by the unique positive root of a certain cubic equation (see (19))[3]. While the study of information transmission and price reactivity yields expected conclusions, the study of indirect utilities results in surprising and novel outcomes that stand in sharp contrast to the related literature.

More precisely, we show (similarly to Kacperczyk et al. [2023]) that insider's price impact makes the market signal fuzzier than in a competitive setting (Proposition 3.1). Insider has a motive to hide part of her signal when submitting her demand and hence she makes the precision of the signal that is revealed by the prices lower. On the other hand, the uniformed traders recognize that the insider reveals a wangled signal and respond with a less elastic demand function, which in turn makes the equilibrium price less reactive to the public information (consistent with the adverse-selection concerns, as in Kyle [1989] and Lou and Rahi [2023]). In fact, these results are robust across the values of all model's parameters. In other words, assuming that the insider is a price taker, we implicitly assume the market gets a more precise signal and prices are more reactive to public information than would be the case if the insider had internalized price impact.

We then turn our attention to traders' indirect utility, which (with a slight abuse of terminology) we refer to as "welfare". Interestingly enough, we get that the insider ex-ante welfare is monotonically increasing in the private signal's precision, when she internalizes her price impact and the uninformed traders are price takers. This result does not hold when the insider trades as price taker[4]. Thus, under the reasonable setting of a strategic insider, price-taker uniformed traders and noisy liquidity providers the meaningful statement "*better information means higher gain*" always holds. Insider's strategic trading increases the value of her private signal making the acquisition of better signal reasonable[5].

In order to sufficiently analyze the effect of price impact on traders' welfare we extensively compare the equilibrium quantities at the equilibria with and without the internalization of insider's price impact. This comparison yields a number of interesting outcomes. For example, we prove analytically that in the absence of private signal, price impact always improves insider welfare. While this is an expected result in ex-ante level, it remarkably holds at any realization of public

[1] As explained in Section 1, assuming price-taker uniformed traders is fairly realistic, since we consider them as the mass of relatively small risk-averse agents who rationally optimize their position but do not have the power to move prices.

[2] Linear price-impact is quite common in the literature (see among others Kyle [1989], Vayanos [1999], Vives [2011]). In fact, the affine structure of price impact is also consistent with the competitive equilibrium, where the combined demand of the insider and the noise traders affects the equilibrium prices in a linear manner. Indeed, the price-taking equilibrium conforms almost exactly to Grossman and Stiglitz [1980] where the equilibrium price is jointly linear in the insider and noise demand.

[3] This stands in contrast to Subrahmanyam [1991] where equilibria are governed by solutions to a quintic equation.

[4] In fact, we provide an example at which when the insider acts as a price taker, her welfare is monotonically decreasing with respect to her signal's precision.

[5] This stands in direct contrast to price taking case of Grossman and Stiglitz [1980] and the case of Nezafat and Schroder [2023] where the uninformed agent internalizes price impact as well.

signal and noise demand. Hence, under symmetric information, the equilibrium at which the insider acts strategically (solely due to her market power) yields higher utility gains for her.

The situation is different under the presence of private signal. While typically (i.e. for most parameters' values) the internalization of price impact yields higher insider's ex-ante welfare, this is not always the case. Indeed, assuming that traders' initial endowments are Pareto-optimally allocated[6], we show that, when the insider is sufficiently risk averse, the uninformed traders are sufficiently risk tolerant and the variance of noise demand is sufficiently low, price impact reduces insider's welfare (the exact condition is given in (37), Proposition 4.6). In order to see why this hold consider a single asset that the insider is expected to buy at equilibrium. Having private information reduces the asset's risk for the insider which in turn increases her expected demand (in line with the empirical evidence of Kacperczyk and Pagnotta [2019]). On the other hand, internalization of price impact makes her hide part of her private signal, which essentially is expected to reduce her demand. This means that price impact and private signal have the opposite expected effect on insider's demand. While equilibrium price decreases due to price impact, the insider's demand is lower under the presence of signal when she is also sufficiently risk averse. If in addition the uniformed traders are highly risk tolerant, their demand increases and hence the insider's trade at market-clearing equilibrium is reduced even further. In other words, under a large deviation of traders' risk aversions, the effect of price impact prevails the one from private signal, insider's demand is reduced and her utility gain is lower. Note that when traders have the same risk aversion, the effect of internalization is always beneficial for the insider.

We conclude that it is the presence of asymmetric information and the deviation on risk aversions that potentially make the price-impact equilibrium welfare reducing for the insider. Details about the above discussion are given in Section 5.

With regards to the above outcomes the following clarification is necessary. As long as the insider internalizes her price impact, equilibrium prices cannot be driven to the corresponding price-taking equilibrium ones. This stems from the uniformed traders' optimal demand. Although they are assumed price takers, when determining their optimal demand they do take into account that the insider internalizes her price impact. As mentioned above, price impact reduces public signal's precision, and hence uniformed traders demand becomes less elastic than in the case of a competitive equilibrium, for any realization of public signal and level of prices. In addition, the lower sensitivity on public signal also affects the intercept point of their demand function (which depends on public signal). Hence, at price-impact equilibrium the insider trades against different residual demand than in the price-taking equilibrium, which essentially implies that price-taking equilibrium cannot be written as a special case of the price-impact one.

Finally, analysis of the uniformed traders' welfare also yields quite interesting outcomes. Under Pareto initial allocation, price-impact equilibrium keeps traders at the same side of trade as in the price-taking equilibrium (say they both take long position). Since insider's price impact decreases the equilibrium prices at that cases, the uniformed traders satisfy their optimal demand but at a discount due to price impact caused by the insider. This holds regardless the presence of private signal, which means that price impact is ex-ante beneficial for the uniformed traders with and without asymmetric information (Proposition 4.5).

[6]Even though traders have CARA preferences, equilibrium quantities under price-impact do depend on the traders' initial endowments. Intuitively, this is because it is insider's total demand, which is based also on her initial risk exposure, that impacts equilibrium prices. Market-clearing condition takes uniformed traders' initial position into account and hence all initial endowments play role at price-impact equilibrium (unless they are optimally allocated before the signal's occurrence). Assuming initial Pareto-allocated endowments turns off the channel where welfare differences arise due to existing hedging demands related to the initial position (independently of private information), and allows us to focus solely on information and internalization effects. While the main discussion of the paper is developed under this assumption, all the equilibrium formulas are stated in the Appendices under a general initial positions.

The above implies that when traders initial endowments are Pareto-optimally allocated, at most of the cases[7] the aggregate welfare of both insider and uniformed traders increases due to the internalization of price impact; a statement that holds with and without the presence of private signal.

Connection with the related literature. The contributions of our paper lie on the on-going literature on price-impact equilibria under asymmetric information, market's welfare and the informativeness of equilibrium prices.

In our price-impact equilibrium, the insider does not act as price-taker. Usually in the related literature the price-taking assumption is primarily made for tractability, because in its absence first one must specify a price impact model, and second, depending upon the specification, it may be very difficult to establish equilibria. In the aforementioned Kyle [1985], Back [1992], Rochet and Vila [1994], insider's demand is combined with exogenous noise traders' demand before being sent to market maker who is risk neutral and prices in a competitive environment. In Subrahmanyam [1991], market makers are allowed to be risk averse while quoting prices to remain at utility indifference (as opposed the uninformed agent of Grossman and Stiglitz [1980] who can be thought of as a market maker who quotes utility-optimal prices), but the insider does not know the noise trader demand before submitting her order (as in Kyle [1985]). Regarding Subrahmanyam [1991] we study in two updating directions. First, by assuming the uninformed agent is a utility optimizer, and second, by allowing the insider to the identify the noise trader demand through the public equilibrium price (as in Rochet and Vila [1994]).

In the spirit of the seminal work of Kyle [1989], several models on normal-CARA setting with price-impact and asymmetric information have been developed. For example, Vayanos [2001] and Rostek and Weretka [2015a] study dynamic thin markets with and without market makers respectively, Rostek and Weretka [2015b] emphasizes on the traders' interdependent preferences and correlated private signals, while Malamud and Rostek [2017] and Anthropelos and Kardaras [2024] consider decentralized exchanges and restricted participation settings respectively. Also, Bergemann et al. [2021] studies a market of divisible goods where agents receive correlated signals and their demand affects the revealed signal at equilibrium, as in our model. In these works, as in Kyle [1989] and Vives [2011], strategic agents submit demand schedule forming a Nash equilibrium. In contrast to our model, they consider all non-noise agents to be strategic, with private information mostly appearing only on investors' endowments. An extensive recent overview of this literature is provided in Rostek and Yoon [2023].

While it is an undoubted fact that large financial institutions invest on obtaining private information of good quality even when trading in markets that are not thin, theoretical studies that justify the positive relation between better information and higher gains from trading are scarce. We highlight that a key factor which always leads to this positive relationship is the insider's market power in a markets with a mass of small uniformed traders and noisy liquidity providers. Under a competitive market setting, the fact that private information has positive value if an investor acts strategically was pointed out back in Hirshleifer [1971], while similar positive effect of private signal on traders' welfare is shown in a competitive model with a continuum of traders in Morris and Shin [2002].

Under non-competitive market settings, information acquisition have been relatively recently studied in Vives [2011], Vives [2014] and Rostek and Weretka [2012], where insiders with correlated noisy signals are considered. An extension of these papers to a two-stage model has been recently developed in Nezafat and Schroder [2023]. Therein, at the first stage, one type of traders can choose the precision of the private signal that they will get before trading. This situation is comparable to our model when the rest of the non-noise traders are uniformed (no private signal). In contrast to our model however, both the insider and uniformed traders are assumed strategic, which essentially implies that the market is thin despite the presence of noise traders. Strategic uniformed traders,

[7] Precisely when inequality (37) does not hold.

together with specific conditions on noise traders' demand, lead to an equilibrium at which private information is welfare-deteriorating. We show that the existence of zero-information equilibria is not possible if the uniformed traders are price takers. In this case, the insider's ex-ante welfare is always increasing with respect to signal's precision, which means that the private information does have positive value for the insider. In addition, we show that under Pareto-allocated initial endowments, private information has positive value for uniformed traders too (even though they are assumed as price-takers).

On the other hand, Kacperczyk et al. [2023] considers strategic informed and price-takers uniformed traders as we do, where the role of initial endowments is highlighted for different types of informed traders. In Gong et al. [2022] the insider is assumed risk neutral and the role of uniformed traders is played by an ambiguous market maker with a quadratic objective. A linear equilibrium, in the spirit of Kyle [1985], is derived where the price is underreacted to public signal, as in our case.

Strategic agents and asymmetric information have been included in Lou and Rahi [2023] too, where in a non-competitive market (in line with Kyle [1989] and Rostek and Weretka [2015a]) traders receive different ex-ante random values of a single asset (and potentially different private signals too). In contrast to our model (and to Kyle [1989]) there is no noise liquidity providers, which essentially implies that price informativeness does not change due to price impact. As in our model, there are conditions that lead to higher ex-ante expected utility for the uniformed traders than the insider. We reach to similar conclusion, but without assuming that uniformed traders act strategically.

Another channel on the valuation of private information that leads to insider's lower gains from trading due to private information is the information sharing. For example, Goldstein et al. [2023] reach to this result in a novel model (based on Kyle [1989]) where informed traders share their private signals before trading (as in Indjejikian et al. [2014]).

Structure of the paper. The rest of the paper is organized as follows. In Section 1, we set up the model and give the necessary initial notation, while in Section 2 we establish the price-impact equilibrium and provide sufficient conditions for its existence. Section 3 develops quantitative analysis and qualitative discussion on information transmission and signal and price sensitivities. Section 4 focuses on welfare comparison at different equilibria and Section 5 is dedicated to equilibria structure regarding prices and risk allocation and concludes with a discussion on model's predictions. All the proofs of the paper are in Appendices, where formulas for more general setting are provided. More precisely, Appendix A deals with proofs of Section 2, Appendix with B those of Sections 3 and 5, while the demanding formulas of Section 4 are proved in Appendix C.

1. Model and Uncertainty

The model has one period. There are d risky assets with terminal payoff $X \sim N(\mu_X, P_X^{-1})$, which we write as

$$X = \mu_X + P_X^{-1/2}\mathcal{E}_X,$$

where $\mathcal{E}_X \sim N(0, 1_d)$. The assets have outstanding supply denoted by the vector $\Pi \in \mathbb{R}^d$. The risk-less asset is in 0 net supply, exogenously priced, and normalized to 1.

Following the related literature (see Subrahmanyam [1991], Spiegel and Subrahmanyam [1992], Grossman [1976], Grossman and Stiglitz [1980] amongst many others), there is an insider I who at time 0 obtains a private signal G, which is a noisy version of X, taking the form

$$G = X + Z_I; \qquad Z_I = P_I^{-1/2}\mathcal{E}_I, \tag{1}$$

where $\mathcal{E}_I \sim N(0, 1_d)$ is independent of X. There is also a mass of uniformed traders who do not receive a private signal, but who in the equilibrium established below, will receive a market signal through the time 0 price. In contrast to Nezafat and Schroder [2023], we assume the uninformed traders are price-takers, and following convention, we consider a representative agent U, hereafter called the uniformed trader. We assume both traders have exponential preferences with respective risk tolerances α_I, α_U. Lastly, there are liquidity providers (also called noise traders), denoted by N, with exogenous demand

$$Z_N = P_N^{-1/2}\mathcal{E}_N,$$

where $\mathcal{E}_n \sim N(0, 1_d)$ is independent of both X and $\mathcal{E}_I$. The matrices P_X, P_I, P_N lie in $\mathbb{S}^d_{++}$, the set of $d \times d$ strictly positive definite symmetric matrices, and $\mu_X \in \mathbb{R}^d$.

Traders I and U are endowed with (constant) share positions $\{\pi_{i,0}\}$ consistent with equilibrium in that $\Pi = \pi_{I,0} + \pi_{U,0}$[8]. At time 0 when the signal arrives, I and U may switch their positions, using their respective information sets, in a self-financing manner to π_I, π_U, which are then held over the period. Writing the to-be-determined equilibrium price as p, the terminal wealth is

$$\mathcal{W}^{\pi_i} := (\pi_{i,0})'p + \pi_i'(X - p); \qquad i \in \{I, U\},$$

where throughout, the symbol $'$ denotes transposition. After the trades, the equilibrium clearing condition is $\Pi = \widehat{\pi}_I + \widehat{\pi}_U + Z_N$, where $\widehat{\pi}_I, \widehat{\pi}_U$ are the optimal positions for I and U and Z_N the noise trader demand. As it is more natural to present results for risk-aversion adjusted strategies, we write

$$\psi_i := \frac{\pi_i}{\alpha_i}; \qquad i \in \{I, U\}.$$

With this notation, the initial and optimal clearing conditions are

$$\Pi = \alpha_I\psi_{I,0} + \alpha_U\psi_{U,0} = \alpha_I\widehat{\psi}_I + \alpha_U\widehat{\psi}_U + Z_N, \tag{2}$$

and the risk-aversion adjusted terminal wealth is

$$\mathcal{W}^{\psi} := \frac{1}{\alpha_i}\mathcal{W}^{\pi} = (\psi_{i,0})'p + \psi'(X - p); \qquad i \in \{I, U\}. \tag{3}$$

Lastly, though it is not required to establish existence of equilibria, at various points below we will assume the initial endowments $\psi_{I,0}, \psi_{U,0}$ are at Pareto optimality given that agents have null information. In other words

Assumption 1.1. *Traders' initial endowments are Pareto optimally allocated in that*

$$\psi_{I,0} = \psi_{U,0} = \widehat{\Pi} := \frac{\Pi}{\alpha_I + \alpha_U}. \tag{4}$$

The idea is that by assuming Pareto optimality prior to the arrival or any signals and noise, we may isolate the effects of differing information and price impact assumptions. Indeed, under this assumption mutually beneficial trading is solely due to signals, noise trader demand, and price impact considerations. Especially in the welfare analysis of Section 4, this assumption greatly simplifies the presentation and allows for easier interpretation of results.

2. The Equilibrium

We now construct the price impact equilibrium. At the end of the section we will also summarize the associated price-taking equilibrium (which is well known in the literature, dating back to Grossman and Stiglitz [1980]) and consider when there is no private information. These latter summaries are made with an eye towards the comparison results of Sections 3 and 4. Proofs of all results are in Appendix A.

Consider when the insider perceives, and hence looks to exploit, her market power, and internalizes her price impact through trading. As the insider's private information is partially revealed

[8] Since the uninformed agent, being a price taker, can be seen as a representative agent for a group of CARA traders, $\pi_{U,0}$ stands for their aggregate initial position and α_U denotes their aggregate risk tolerance. Also, one may allow the liquidity provides to have initial endowment $\pi_{N,0} \neq 0$, but this could just be absorbed into the supply Π. As such, we take $\pi_{N,0} = 0$.

to the market through her demand (which affects the uniformed trader's optimal demand and hence the equilibrium price), by accounting for her ability to impact the market, she controls the signal revealed to the market. Therefore we expect the insider's internalization of impact to affect not only equilibrium prices, but also equilibrium information transmission. By contrast, we assume the uniformed agent takes prices, and hence time 0 information, as given. We believe this setting is reasonable, as the uniformed trader is in fact a representative agent for a mass of "small" uniformed traders who do not have the power to impact the market.

As is common in the literature, we seek a linear impact equilibrium. In other words, the insider perceives that if she changes her position from $\pi_{I,0} = \alpha_I \psi_{I,0}$ to $\pi_I = \alpha_I \psi_I$, then the price will be an affine function of her trade combined with the noise trader demand,

$$p_\iota(\psi, Z_N) = V + M\left(\psi_I - \psi_{I,0} + \frac{Z_N}{\alpha_I}\right), \tag{5}$$

for a vector V and a matrix M that are to-be-determined in equilibrium[9], and where throughout we use the subscript "ι" to stand for "impact". Next, following the analysis of Rochet and Vila [1994] we assume the insider can see both her private signal and, for a given trade ψ, the price p_ι of (5). This will imply that the noise trader demand Z_N is revealed to the insider through her signal and the time 0 price. This is in contrast to Spiegel and Subrahmanyam [1992], Subrahmanyam [1991] and leads to a different equilibrium, but we believe it is a realistic assumption, given that the equilibrium price (as will be shown) is linear in the signal and noise.

To make this assumption precise, note that G and Z_N are the only random quantities revealed at time 0, and hence every insider strategy must be known using the information generated by G and Z_N. Therefore, if the insider uses a strategy ψ which reveals the noise trader demand Z_N through the price, it must also be that Z_N is known given the information generated by G and $p_\iota(\psi, Z_N)$. As such, we define set of acceptable trading strategies for the insider to be

$$\mathcal{A}_I := \{\psi \in \sigma(G, Z_N) \mid Z_N \in \sigma(G, p_\iota(\psi, Z_N))\}^{10}.$$

Here, "$\psi \in \sigma(G, H)$" means that ψ is $\sigma(G, H)$ measurable, and note that for any $\psi \in \mathcal{A}_I$, one has $\sigma(G, p_\iota(\psi, Z_N)) = \sigma(G, Z_N)$.

As the insider knows the noise demand, the public quantity $\psi - \psi_{I,0} + Z_N/\alpha_I$ is (partially) controlled by insider, and its effect on equilibrium can be understood as her price impact. Indeed, when the uniformed trader acts optimally as price taker, the price will take the affine form in (5). As the price is public, the combined insider and noise trader demand changes the public information set, effectively altering the precision of a public signal (taking the same form as in (1) just with a different noise term). Therefore, when the insider accounts for her price impact, the uniformed trader also recognizes that the precision of the public signal will change with the insider's policy. This is in contrast to the price-taking case and in fact implies the public signal is always of a lower quality when the insider internalizes impact (see Proposition 3.1).

Our first goal is to identify a vector V and matrix M which clears the market. The first step in doing so is to characterize the insider's optimal demand for any fixed M and V. To this end, the insider's optimal investment problem is

$$\inf_{\psi \in \mathcal{A}_I} \mathbb{E}\left[e^{-\left(\psi_{I,0}' p_\iota(\psi, Z_N) + \psi'(X - p_\iota(\psi, Z_N))\right)} \middle| \sigma(G, Z_N)\right]. \tag{6}$$

To identify the optimal $\widehat{\psi}_I$, we write $P_{X|G}$ as the precision of X given the insider signal G, and (because it is technically useful) we express M in terms of $P_{X|G}$ and a to-be-determined matrix $\mathcal{Y}$.

$$P_{X|G} := P_I + P_X; \qquad M = M(\mathcal{Y}) = P_{X|G}^{-1/2}\ \mathcal{Y}\ P_{X|G}^{-1/2}. \tag{7}$$

[9]The affine structure could also be deduced by the insider if she first examined the equilibrium structure in the price taking case - see Remark 2.10 below. However, as affine impact is so common, we take it as a primitive.

[10]We will show the optimal strategy among all $\sigma(G, Z_N)$ measurable policies lies in $\mathcal{A}_I$, so $\mathcal{A}_I$ poses no restriction. However, if one does not restrict to $\mathcal{A}_I$ a-priori, it is not clear how to obtain the law of X conditional on $\sigma(G, p_\iota(\psi, Z_N))$.

The dependence of M on $\mathcal{Y}$ will force V to depend on $\mathcal{Y}$ as well, so we write $V = V(\mathcal{Y})$. As we will see, $1 + \mathcal{Y} + \mathcal{Y}' \in \mathbb{S}^d_{++}$ ensures well-posedness of the minimization problem (6) but realistically we expect a positive impact function (i.e. $M + M' \in \mathbb{S}^d_{++}$) so that $\mathcal{Y} + \mathcal{Y}' \in \mathbb{S}^d_{++}$. The following lemma identifies the insider's optimal demand in terms of matrix $\mathcal{Y}$.

Lemma 2.1. *Let $g, z \in \mathbb{R}$. On the set $\{G = g, Z_N = z\}$, the unique optimizer of* (6) *enforces*

$$\widehat{\psi}_{I,\iota}(g,z) - \psi_{I,0} + \frac{z}{\alpha_I} = \mathcal{M}\left(g + \Lambda_\iota z + \mathcal{V}\right), \tag{8}$$

where

$$\begin{aligned} \mathcal{M} = \mathcal{M}(\mathcal{Y}) &:= P_{X|G}^{1/2}(1_d + \mathcal{Y} + \mathcal{Y}')^{-1} P_{X|G}^{-1/2} P_I, \\ \Lambda_\iota = \Lambda_\iota(\mathcal{Y}) &:= \frac{1}{\alpha_I} P_I^{-1} P_{X|G}^{1/2}(1_d + \mathcal{Y}') P_{X|G}^{-1/2}, \\ \mathcal{V} = \mathcal{V}(\mathcal{Y}) &:= P_I^{-1}\left(P_X \mu_X - \psi_{I,0} - P_{X|G} V(\mathcal{Y})\right). \end{aligned} \tag{9}$$

Remark 2.2. $\widehat{\psi}_{I,\iota}$ is in fact optimal among all functions ψ (not just those in $\mathcal{A}_I$). This is because $\widehat{\psi}_{I,\iota}$ is affine in z with matrix coefficient C such that $C + 1_d/\alpha_I$ is invertible. Therefore, the class $\mathcal{A}_I$ poses no restriction.

The next step is to identify the uninformed trader's optimal demand, which first involves describing his time 0 information. The clearing condition (2) implies that in equilibrium $\widehat{\psi}_{I,\iota} - \psi_{I,0} + Z_N/\alpha_I$ must be publicly observable, and using (8) it is natural to define the market signal

$$H_\iota := G + \Lambda_\iota Z_N = X + Z_I + \Lambda_\iota Z_N, \tag{10}$$

which is of the same form as the insider signal G, except with lower precision

$$P_{U,\iota} = P_{U,\iota}(\mathcal{Y}) := \left(P_I^{-1} + \Lambda_\iota(\mathcal{Y}) P_N^{-1} \Lambda_\iota'(\mathcal{Y})\right)^{-1}. \tag{11}$$

By observing the price, the uniformed trader has time 0 information $\sigma(H_\iota)$ and written as a function of H_ι, the price is $p(H_\iota)$ where

$$p_\iota(h_\iota) = M\mathcal{M}(h_\iota + \mathcal{V}) + V, \tag{12}$$

which we will simplify below. As the uninformed is a price taker, his optimization problem is

$$\inf_{\psi \in \sigma(H_\iota)} \mathbb{E}\left[e^{-\left(\psi_{U,0}' p_\iota(H_\iota) + \psi'(X - p_\iota(H_\iota))\right)} \middle| \sigma(H_\iota)\right].$$

Similarly to Lemma 2.1 we obtain

Lemma 2.3. *Let $h_\iota \in \mathbb{R}$. On the set $\{H_\iota = h_\iota\}$, the uninformed agent has risk-aversion adjusted optimal demand*

$$\widehat{\psi}_{U,\iota}(h_\iota) = P_{U,\iota} h_\iota + P_X \mu_X - (P_{U,\iota} + P_X) p_\iota(h_\iota). \tag{13}$$

Relations (10)-(13) quantify the insider's impact on the market signal H_ι, the market signal precision $P_{U,\iota}$, the price $p_\iota(H_\iota)$, and the uninformed trader's optimal demand $\widehat{\psi}_{U,\iota}$. These impacts are not consistent with the price taking equilibrium discussed below. When the insider internalizes her price impact, the uniformed trader takes it into account, and the coefficients of her affine demand function change (in fact, the reactions to the public signal and its precision are lower as we show in Section 3). This means that even when the insider submits her optimal price-impact demand in the price taking equilibrium, the market will not equilibrate to the price-impact price, as the uniformed trader's optimal demand alters. We return to this point in Remark 2.11.

The last step is to identify $\mathcal{Y}$ and $V(\mathcal{Y})$ by enforcing the market-clearing condition (2),

$$\Pi = \alpha_I\left(\widehat{\psi}_{I,\iota}(G, Z_N) - \psi_{I,0} + \frac{Z_N}{\alpha_I}\right) + \alpha_I \psi_{I,0} + \alpha_U \widehat{\psi}_{U,\iota}(H_\iota). \tag{14}$$

Taking into account (8), (9) and (13), the above clearing condition will induce the equilibrium conditions that $\mathcal{Y}$ and V should satisfy. In fact, (positive) solutions to a matrix-valued cubic equation for $\mathcal{Y}$ are in one-to-one correspondence with equilibrium, as the following shows. To state the proposition, recall $\widehat{\Pi}$ from (4) and define

$$p_0 := \mu_X - P_X^{-1}\widehat{\Pi}. \tag{15}$$

This is the equilibrium price corresponding to the (no private information, no price impact) Pareto optimal initial endowments of Assumption 1.1, and it provides an intuitive way to gauge how private information and price impact alter prices.

Proposition 2.4. *Assume $\widehat{\mathcal{Y}}$ enforces the matrix equality*

$$0_d = P_{U,\iota}(\widehat{\mathcal{Y}})\mathcal{M}(\mathcal{Y})^{-1} + \frac{\alpha_I}{\alpha_U}1_d - (P_{U,\iota}(\widehat{\mathcal{Y}}) + P_X)M(\widehat{\mathcal{Y}}), \tag{16}$$

where M, $\mathcal{M}$ and $P_{U,\iota}$ are from (7), (9) *and* (11). *Then, there exists a price-impact equilibrium. The market signal is H_ι from* (10), *and the equilibrium price p_ι is of the form* (12) *with $M = M(\widehat{\mathcal{Y}})$ from* (7), *and*

$$V = V(\widehat{\mathcal{Y}}) = p_0 + P_X^{-1}P_\iota(\widehat{\mathcal{Y}})(P_I - P_\iota(\widehat{\mathcal{Y}}))^{-1}\left(\psi_{I,0} - \widehat{\Pi}\right); \qquad P_\iota(\widehat{\mathcal{Y}}) := \frac{\alpha_I P_I + \alpha_U P_{U,\iota}(\widehat{\mathcal{Y}})}{\alpha_I + \alpha_U}. \tag{17}$$

Lastly, the price function p_ι takes the form

$$\begin{aligned} p_\iota(h_\iota) &= p_0 + M(\widehat{\mathcal{Y}})\mathcal{M}(\widehat{\mathcal{Y}})\,(h_\iota - p_0) \\ &\quad + \left(P_X^{-1} - M(\widehat{\mathcal{Y}})\mathcal{M}(\widehat{\mathcal{Y}})\left(P_\iota(\widehat{\mathcal{Y}})^{-1} + P_X^{-1}\right)\right)P_\iota(\widehat{\mathcal{Y}})(P_I - P_\iota(\widehat{\mathcal{Y}}))^{-1}\left(\psi_{I,0} - \widehat{\Pi}\right). \end{aligned} \tag{18}$$

Remark 2.5. When the initial endowments satisfy Assumption 1.1 we obtain the simple expression $p_\iota(H_\iota) = p_0 + M(\widehat{\mathcal{Y}})\mathcal{M}(\widehat{\mathcal{Y}})(H_\iota - p_0)$, which explicitly identifies the pricing change due to private information and price impact as $M(\widehat{\mathcal{Y}})\mathcal{M}(\widehat{\mathcal{Y}})(H_\iota - p_0)$. Note also that

$$p_0 = \mu_X - P_X^{-1}\widehat{\Pi} = \mathbb{E}^{\mathbb{Q}_0}[X]; \qquad \frac{d\mathbb{Q}_0}{d\mathbb{P}} = \frac{e^{-\widehat{\Pi}'X}}{\mathbb{E}\left[e^{-\widehat{\Pi}'X}\right]}.$$

Here, $\mathbb{Q}_0$ is the pricing measure associated to the (no private information, no price impact) Pareto optimal initial endowments of Assumption 1.1. As X is $\mathbb{P}$ independent of the noise terms Z_I, Z_N in (10), it follows that $p_0 = \mathbb{E}^{\mathbb{Q}_0}[H_\iota] = \mathbb{E}^{\mathbb{Q}_0}[p(H_\iota)]$ so that on average under $\mathbb{Q}_0$ the prices with price impact/private information and without price impact/private information coincide.

In light of Proposition 2.4, our goal is to find solutions $\widehat{\mathcal{Y}}$ to (16). This is a matrix-valued cubic equation for $\mathcal{Y}$[11], and the primary difficulty in establishing existence of solutions $\widehat{\mathcal{Y}}$ arises due to the interaction between the precision matrices P_I, P_N and P_X. While the general case appears intractable, we are able to obtain existence of an equilibrium under a simplifying assumption, which is always valid in the case of a single asset.

Assumption 2.6. *$P_I = p_I P_X$ and $P_N = p_N P_X^{-1}$ for scalars $p_I, p_N > 0$.*

Under this assumption we have the following result

Proposition 2.7. *Let Assumption 2.6 hold. Then there exists a solution $\widehat{\mathcal{Y}}$ to* (16) *which takes the form $\widehat{\mathcal{Y}} = \widehat{y}1_d$ where $\widehat{y}$ is the unique positive solution of the cubic equation*

$$0 = (1+y)^2\left(1 - \frac{\alpha_U y}{\alpha_I(1+p_I)}\right) + \alpha_I^2 p_N p_I\left(\frac{\alpha_U}{\alpha_I}(1+y) + 1\right). \tag{19}$$

[11] This differs from the quintic equation in Subrahmanyam [1991] when the insider does not see the noise trader demand.

The positivity of $\widehat{y}$ is desirable because from (7) it implies $M \in \mathbb{S}^d_{++}$. This ensures the pricing function is increasing in the combined trade $\widehat{\psi}_{I,\iota} - \psi_{I,0} + Z_N/\alpha_I$. Additionally, as it solves a cubic equation, in principle, there is an explicit formula for $\widehat{y}$ which may be analyzed. From Proposition 2.7 we immediately obtain the main result of the section.

Theorem 2.8. *Let Assumption 2.6 hold. Then a price-impact equilibrium exists. Using $\widehat{y}$ from Proposition 2.7 and $\mathcal{Y} = \widehat{y}1_d$, the market signal is H_ι from* (10)*. The equilibrium price is $p_\iota(H_\iota)$ for p_ι from* (18)*. The insider uses the optimal policy $\widehat{\pi}_{I,\iota} = (1/\alpha_I)\widehat{\psi}_{I,\iota}(G, H_\iota)$ for $\widehat{\psi}_{I,\iota}$ from* (8)*. The uninformed agent uses the optimal policy $\widehat{\pi}_{U,\iota} = (1/\alpha_U)\widehat{\psi}_{U,\iota}(H)$ for $\widehat{\psi}_{U,\iota}$ from* (13)*.*

Price-taking equilibrium. For comparison purposes (i.e. to provide a benchmark case) in this section we consider when all traders are price takers. As this result is well known (see Grossman and Stiglitz [1980]), we summarize the equilibrium structure in the following proposition, a proof of which is given in Appendix A. To state the proposition, assume there is a market signal H revealed through the time 0 price $p = p(H)$, and both traders take $p(H)$ as given. The insider has time 0 information $\sigma(H, G)$ while the uninformed trader uses $\sigma(H)$. Using (3), the insider and uninformed trader's optimal investment problems are respectively

$$\inf_{\psi \in \sigma(G,H)} \mathbb{E}\left[e^{-\mathcal{W}^\psi} \middle| \sigma(G, H)\right]; \qquad \inf_{\psi \in \sigma(H)} \mathbb{E}\left[e^{-\mathcal{W}^\psi} \middle| \sigma(H)\right]. \tag{20}$$

We say $(H, p(H))$ is a price-taking equilibrium if the clearing condition (2) holds for the resultant optimal policies.

Proposition 2.9. *There is a price-taking equilibrium. The market signal is*

$$H := G + \frac{1}{\alpha_I} P_I^{-1} Z_N = X + Z_I + \frac{1}{\alpha_I} P_I^{-1} Z_N. \tag{21}$$

H is of the same form as G, but with lower precision

$$P_U := \left(P_I^{-1} + \frac{1}{\alpha_I^2} P_I^{-1} P_N^{-1} P_I^{-1}\right)^{-1}. \tag{22}$$

With p_0 from (15)*, the equilibrium price is $p = p(H)$ for the price function*

$$p(h) := p_0 + (P_X + P)^{-1} P (h - p_0); \quad P := \frac{\alpha_I P_I + \alpha_U P_U}{\alpha_I + \alpha_U}. \tag{23}$$

The optimal policies for I and U are $\widehat{\psi}_I(G, H)$ and $\widehat{\psi}_U(H)$ respectively, where

$$\widehat{\psi}_I(g, h) = P_X\mu_X + P_I g - (P_I + P_X)p(h); \quad \widehat{\psi}_U(h) = P_X\mu_X + P_U h - (P_U + P_X)p(h). \tag{24}$$

Remark 2.10. As mentioned above, we directly assumed linear price impact. This assumption is motivated by the fact that in the price-taking case, the impact is indeed linear. To see this, note that on the set $\{G = g, Z_N = z\}$, the combined trade of the insider and noise trader is

$$\widehat{\psi}_I(g, z) - \psi_{I,0} + \frac{z}{\alpha_I} = P_X\mu_X + P_I h(g, z) - \psi_{I,0} - (P_I + P_X)p(h(g, z)).$$

The above combined demand function and the corresponding uniformed trader's optimal demand yields the clears out the market at the price (23). Expressing the signal in terms of the price we obtain $h = p_0 + P^{-1}(P_X + P)(p(h) - p_0)$, and after some simple algebra one has

$$p(g, z) = p_0 + P_X^{-1} P (P_I - P)^{-1} \left(\widehat{\psi}_I(g, z) + \frac{z}{\alpha_I} - \widehat{\Pi}\right). \tag{25}$$

The above is the reverse combined demand function at equilibrium, and indicates the linearity of price impact. Indeed, even though the insider does not internalize impact in the price-taking case, in equilibrium it turns out the price is linearly impacted by her trade, combined with the noise trader's demand. The price takes the form (5), where the vector V and a matrix M are

$$M = P_X^{-1} P (P_I - P)^{-1}; \qquad V = p_0 + P_X^{-1} P (P_I - P)^{-1} \left(\psi_{I,0} - \widehat{\Pi} \right). \tag{26}$$

Remark 2.11. As we have already discussed, as long as the insider internalizes her price impact and the uniformed trader takes it into account, the price taking and price impact equilibria cannot coincide. In fact, even if the insider submits her price taking equilibrium demand when internalizing impact, the market will not equilibrate to the price-taking equilibrium price. This would be the case if the uniformed trader did not perceive the change in market signal precision due to the insider's demand (i.e., if he ignored the insider's internalization of the price impact). Indeed, if $P_{U,\iota}(\mathcal{Y}) = P_U$ for all $\mathcal{Y}$ then (17) implies that V is the same as in (26).

On the other hand, there is a $\mathcal{Y}^* \in \mathbb{S}^d_{++}$ such that $M(\mathcal{Y}^*) = P_X^{-1} P (P_I - P)^{-1}$ (see (26)). For this $M(\mathcal{Y}^*)$, if the insider used the price taking optimal demand $\widehat{\psi}_I$ from (24), it would reveal the same signal to the uniformed trader as in the price-taking equilibrium. This would lead to the same uniformed trader's demand and hence the same clearing price as in price-taking equilibrium. However, when M and V are as in (26), $\widehat{\psi}_I$ from (24) is not optimal for the insider. This is because (9), which identifies the optimal demand under any linear price impact is not consistent with (26).

No private signal (NS) equilibrium. The price-taking results of the previous section allow us to isolate the effects of price-impact in the presence of information asymmetry. In this section, we turn off the asymmetric information channel to analyze the effects due solely to internalization of price impact. We envision a situation where there is a market maker who is capable of moving prices, but who is not privately informed about the asset's terminal payoff. She wants to move prices against a mass of (small) uninformed traders in a way to maximize her utility. Now, to formally establish equilibrium, one would have to repeat the analysis of Section 2 removing the private signal, giving all agents the same information set. However, it turns out that under Assumption 2.6, the no-signal equilibrium (in both the price-impact and price-taking cases) coincides with the previously established equilibrium in the limit $p_I \to 0$. The resultant equilibrium prices and optimal positions are summarized in the following proposition, the proof of which is given in Section A below. To state it, recall $\widehat{\Pi}$ and p_0 from (4) and (15).

Proposition 2.12. *The no-signal equilibrium corresponds to $p_I = 0$. In the price-taking case, the equilibrium price and the risk aversion adjusted optimal positions are $p_{ns}(Z_N)$ and $\widehat{\psi}_{ns,I} = \widehat{\psi}_{ns,U} = \widehat{\psi}_{ns}(Z_N)$ where*

$$p_{ns}(z) = p_0 + \lambda P_X^{-1} \frac{z}{\alpha_I}; \qquad \widehat{\psi}_{ns}(z) = \widehat{\Pi} - \lambda \frac{z}{\alpha_I}, \tag{27}$$

where $\lambda = \alpha_I/(\alpha_I + \alpha_U)$ is the insider's proportion of the total risk tolerance. In the price-impact case, the equilibrium price is $p_{ns,\iota}(Z_N)$ where

$$p_{ns,\iota}(z) = p_0 + \frac{\lambda}{1-\lambda^2} P_X^{-1} \frac{z}{\alpha_I} + \frac{\lambda^2}{1-\lambda^2} P_X^{-1} \left(\psi_{I,0} - \widehat{\Pi} \right).$$

The risk aversion adjusted optimal policies for I and U are $\widehat{\psi}_{ns,\iota,I}(Z_N)$ and $\widehat{\psi}_{ns,\iota,U}(Z_N)$, where

$$\widehat{\psi}_{ns,\iota,I}(z) = \psi_{I,0} - \frac{\lambda}{1+\lambda} \frac{z}{\alpha_I} - \frac{1}{1+\lambda} \left(\psi_{I,0} - \widehat{\Pi} \right),$$

$$\widehat{\psi}_{ns,\iota,U}(z) = \psi_{I,0} - \frac{\lambda}{1-\lambda^2} \frac{z}{\alpha_I} - \frac{1}{1-\lambda^2} \left(\psi_{I,0} - \widehat{\Pi} \right).$$

3. Comparison Analysis: Signals and Price Sensitivity

In this section, we focuses on the comparison of quality of the public signals and price sensitivity with respect to signals. We label the price impact equilibria as "PI" and the price taking equilibria as "PT". We show PI public signal is of a worse quality, and prices are less responsive to not only the market and insider signals, but also to the publicly observable (risk-tolerance weighted) insider's and noise trader residual demand $\widehat{\psi}_I - \pi_{I,0} + Z_N/\alpha_I$. As such, the main message of this section is

> *By assuming the insider is a price taker, one overestimates the quality of the public signal and the reactivity of equilibrium prices.*

Throughout, we enforce Assumption 2.6, include the subscript ι when describing any quantity obtained internalizing price impact. Proofs of all results here are in Appendix B. Additionally, we define (see Proposition 2.12)

$$\kappa := \alpha_I^2 p_N; \qquad \lambda := \frac{\alpha_I}{\alpha_I + \alpha_U}, \tag{28}$$

as with this notation, (19) becomes

$$0 = (1+y)^2\left(1 - \frac{1-\lambda}{\lambda(1+p_I)}y\right) + \frac{\kappa p_I}{\lambda}\left((1-\lambda)y + 1\right). \tag{29}$$

Lastly, when $\mathcal{Y} = y1_d$ the quantities in (7), (9) and (11) take the form

$$M = \frac{y}{1+p_I}P_X^{-1}, \quad \mathcal{M} = \frac{p_I}{1+2y}P_X, \quad \Lambda_\iota = \frac{y+1}{\alpha_I p_I}P_X^{-1}, \quad P_{U,\iota} = \left(\frac{\kappa p_I^2}{(1+y)^2 + \kappa p_I}\right)P_X. \tag{30}$$

Signal quality. As we have seen, in both the PI and PT equilibria a signal of the form "$X+$Noise" is communicated to market. It is natural to ask which signal is of a higher quality, or even more pointedly, is the public signal less informative in the presence of price impact? To address these questions we write the market signals as functions of the insider signal G and noisy demand Z_N, and using (10), (21) and (30) we obtain

$$h_\iota(g,z) = g + \frac{1+\widehat{y}}{p_I}P_X^{-1}\frac{z}{\alpha_I}; \qquad h(g,z) = g + \frac{1}{p_I}P_X^{-1}\frac{z}{\alpha_I}. \tag{31}$$

From Proposition 2.7 we know $\widehat{y} > 0$, which implies the following result.

Proposition 3.1. *The market signal is noisier when the insider accounts for price impact. In fact,* $P_U > P_{U,\iota}$.

The fact that the public signal is noisier (i.e. less informative) under price impact is associated with the way the uniformed trader determines his demand. As already mentioned, the uniformed trader accounts for the insider's internalization of price impact, and in his optimization problem considers $P_{U,\iota}$ instead of P_U (see the demand functions (13) and (24)). In other words, he recognizes the insider reveals a wangled signal, and responds with a less elastic demand function.

Remark 3.2. Accounting for price impact changes the public signal in the direction of the noise traders' (liquidity providers') order. As $\widehat{y} > 0$, we have $(h_\iota(g,z) - h(g,z))'z \geq 0$ for all (g,z). In words, the strategically revealed signal by the insider is higher if and only if there is positive demand from the noise traders. As we will analyze in Section 5, under Assumption 1.1, positive z implies that the insider and uniformed trader take short position at the equilibrium and hence the strategically enhanced public signal increases the prices that the traders sell the assets to the liquidity providers.

Price reactivity. In both the PI and PT equilibria, prices are affine functions of the respective public signals. However, the coefficients in the functions differ. This leads one to ask whether price impact increases or decreases the sensitivity of prices with respect to public signaling. To answer this, we first re-express the prices from (18) and (23) using the notation of (15), (28) and (30).

Proposition 3.3. *The pricing functions take the form*

$$
\begin{aligned}
p_\iota(h_\iota) &= p_0 + \frac{p_I \widehat{y}}{(1+p_I)(1+2\widehat{y})} \times (h_\iota - p_0) + \frac{\lambda \widehat{y}\left(\kappa p_I + (1+\widehat{y})^2\right)}{(1-\lambda)(1+2\widehat{y})(1+\widehat{y})^2} \times P_X^{-1}\left(\psi_{I,0} - \widehat{\Pi}\right), \\
p(h) &= p_0 + \frac{p_I(\kappa p_I + \lambda)}{1-\lambda+(1+p_I)(\kappa p_I+\lambda)} \times (h - p_0).
\end{aligned} \tag{32}
$$

Given this, we now consider reactivity to the market and, equivalently in view of (31), insider signals. To do so, define the slopes

$$
m_{g,\iota} = \frac{p_I \widehat{y}}{(1+p_I)(1+2\widehat{y})}, \qquad m_g = \frac{p_I(\kappa p_I+\lambda)}{1-\lambda+(1+p_I)(\kappa p_I+\lambda)}. \tag{33}
$$

The following proposition shows that prices are always more reactive to the insider, and hence to the market signal, when the insider does not internalize price impact. This is directly linked with the lower elasticity of the uniformed trader's demand function due to price impact. Note that this is an endogenously derived outcome. The insider has a motive to make the public signal noisier, which in turn makes uniformed trader less elastic and yields prices which are less sensitive to the public signal.

Proposition 3.4. *The equilibrium prices are less sensitive to the market signal when the insider accounts for price impact, i.e.* $m_{g,\iota} < m_g$.

We conclude this discussion with the price reactivity with respect to the publicly observable (weighted risk-tolerance adjusted) combined demand

$$
\widehat{\chi}_\iota := \widehat{\psi}_{I,\iota}(G, Z_N) - \psi_{I,0} + \frac{1}{\alpha_I} Z_N, \qquad \widehat{\chi} := \widehat{\psi}_I(G, Z_N) - \psi_{I,0} + \frac{1}{\alpha_I} Z_N.
$$

Using (5) and (25), prices are affine in the combined demand with respective slopes

$$
m_{\widehat{\chi},\iota} = \widehat{y}\,(P_I + P_X)^{-1}, \qquad m_{\widehat{\chi}} = P_X^{-1} P (P_I - P)^{-1},
$$

where we have used (7), and where P is from (22) and (23). As expected from the preceding analysis, when the insider internalizes her impact, prices are less sensitive to the publicly observable combined demand (similarly to the public signal). Again, we stress this is an endogenous outcome, arising from the insider's strategy when she internalizes her price impact. The next proposition formally states this result.

Proposition 3.5. *The prices are less sensitive to the publicly observable combined demand when the insider accounts for price impact, i.e.* $m_{\widehat{\chi},\iota} < m_{\widehat{\chi}}$.

4. Welfare Analysis

Overview. This section is dedicated to analyzing the traders' welfare. We primarily study two issues: how the insider's signal quality is translated to her welfare from trading; and the effect of price-impact internalization on traders' welfare. We again lable the price impact equilibrium PI and the price taking equilibrium PT.

Following Laffont [1985], we define welfare at both the ex-ante level (i.e. at time $0-$, prior to signal revelation) and interim level (at time 0, after the signal revelation) in terms of certainty equivalents. As such one can alternatively think of this comparison as a comparison of indirect utility gains from trading. We use the subscripts $0-$ and 0 to indicate ex-ante and interim welfare respectively. Welfare will always be computed using the overall wealth in (3). Given this, we denote by $\widehat{\mathcal{W}}_{I,\iota}, \widehat{\mathcal{W}}_{U,\iota}$ the optimal terminal wealths in the PI equilibrium, and $\widehat{\mathcal{W}}_I, \widehat{\mathcal{W}}_U$ those in the PT case. Then, for $k \in \{\ ,\iota\}$ the corresponding interim certainty equivalents are

$$
\begin{aligned}
\mathrm{CE}^I_{0,k} &= -\alpha_I \log\left(\mathbb{E}\left[e^{-(1/\alpha_I)\widehat{\mathcal{W}}_{I,k}} \middle| \sigma(G, H_k)\right]\right), \\
\mathrm{CE}^U_{0,k} &= -\alpha_U \log\left(\mathbb{E}\left[e^{-(1/\alpha_U)\widehat{\mathcal{W}}_{U,k}} \middle| \sigma(H_k)\right]\right),
\end{aligned}
$$

while the ex-ante certainty equivalents are

$$\mathrm{CE}^{j}_{0-,k} = -\alpha_j \log\left(\mathbb{E}\left[e^{-(1/\alpha_j)\widehat{\mathcal{W}}_{j,k}}\right]\right); \qquad j \in \{I, U\}.$$

To emphasize the effects of private signal quality and price impact internalization on certainty equivalents, throughout (except stated otherwise) we impose both Assumption 2.6 and the initial Pareto allocation in Assumption 1.1, which implies that initially, the insider and uninformed traders have identical risk-aversion adjusted risk exposure. Our main findings are summarized below.

First, in the PI equilibrium, insider welfare is monotonically increasing in the precision p_I. As mentioned in the introductory section, this is not a standard result, as in the majority of private-information models, better precision does not always imply higher utility gains. For instance, this result stands in direct contrast to Nezafat and Schroder [2023], and shows the conclusions there-in are a consequence of assuming the uninformed trader also internalizes his impact on prices. Additionally, this result is also in contrast to Grossman and Stiglitz [1980], where all traders are price takers. Therefore, one only obtains the reasonable conclusion that a better signal is better for the insider (if not, why would the insider expend effort to obtain the signal?) when there is a differential between the insider and uniformed traders, not only in terms of information, but also in terms of their internalization of price impact.

Second, insider ex-ante welfare is not always higher in the PI equilibrium. While typically this is true, as the insider becomes increasingly more risk averse, PT welfare will exceed PI welfare when also the uninformed trader has sufficiently high risk tolerance, and provided the insider signal quality is not too low. However, in contrast to the insider, uninformed ex-ante welfare is always higher in the PI case.

Third, absent private information, insider welfare is always higher in the PI case. Remarkably, this holds at the interim level, and does not require the initial allocation to be Pareto (i.e. Assumption 1.1). There is no corresponding statement for uniformed trader as interim welfare may be higher or lower. However, if Assumption 1.1 holds, then in the no-signal case, we can order interim welfare as follows

$$\mathrm{U(PI)} > \mathrm{I(PI)} > \mathrm{U(PT)} = \mathrm{I(PT)},$$

so that uninformed trader's welfare exceeds insider's welfare in the price-impact case. We use the notation in (28) and, aside from Proposition 4.4, all proofs in this section are in Appendix D.

Certainty Equivalents. We start by calculating the certainty equivalents. Propositions C.4 and C.5 explicitly compute ex-ante welfare for both types of traders in the two price-impact and price-taking equilibria. While the formulas there-in are very long, a significant simplification occurs when the initial allocation is Pareto, which we now give. To state the proposition define

$$\mathrm{CE}^{i}_{nsn} := \alpha_i\left(\widehat{\Pi}'\mu_X - \frac{1}{2}\widehat{\Pi}'P_X^{-1}\widehat{\Pi}\right), \qquad \text{for } i = I, U, \tag{34}$$

as the certainty equivalents associated to the Pareto initial allocations. With this notation, under Assumption 1.1, Propositions C.4 and C.5 in Appendix C, along with the notation in (28) yield the following.

Proposition 4.1. *Let Assumptions 1.1 and 2.6 hold. In the PI equilibrium, with $\widehat{y}$ from* (29)*,*

$$CE^{I}_{0-,\iota} = CE^{I}_{nsn} + \alpha_I \frac{d}{2}\log\left(1 + \frac{\kappa p_I(1+p_I) + \widehat{y}^2}{\kappa(1+p_I)(1+2\widehat{y})}\right),$$
$$CE^{U}_{0-,\iota} = CE^{U}_{nsn} + \alpha_U \frac{d}{2}\log\left(1 + \frac{\lambda^2(\kappa p_I + (1+\widehat{y})^2)}{(1-\lambda)^2\kappa(1+2\widehat{y})^2}\right).$$

In the PT equilibrium

$$CE^I_{0-} = CE^I_{nsn} + \alpha_I \frac{d}{2} \log\left(1 + \frac{(1-\lambda)^2\kappa p + (1+p_I)(\lambda + \kappa p_I)^2}{\kappa(1+\lambda p_I + \kappa p_I(1+p_I))^2}\right),$$
$$CE^U_{0-} = CE^U_{nsn} + \alpha_U \frac{d}{2} \log\left(1 + \frac{\lambda^2(1+\kappa p_I)}{\kappa(1+\lambda p_I + \kappa p_I(1+p_I))^2}\right).$$

Welfare and the insider's signal precision. Here we analyze insider welfare with respect to the signal precision p_I. Though not modeled, it presumably costs effort, time and/or money to both obtain and refine the signal. In fact, one of the most important questions in the related literature is whether it is worth for the traders who have the ability and resources to obtain a private signal to actually pay the cost and obtain it. As such, one should examine whether the benefits of the private signal (as measured by welfare) are increasing with respect to the quality of a signal (as measured by signal precision). This is of course connected with the related cost, in the sense that better signal is normally linked to higher cost.

In the PI equilibrium, using Proposition 4.1 for fixed κ, λ, it suffices to study the map

$$p_I \to \phi_\iota(p_I) := \frac{\kappa p_I(1+p_I) + \widehat{y}(p_I)^2}{\kappa(1+p_I)(1+2\widehat{y}(p_I))}, \tag{35}$$

where $\widehat{y} = \widehat{y}(p_I)$ is the unique positive solution of (29). Numerically, this can easily seen to be increasing in p_I by randomly sampling $\kappa > 0$, $\lambda \in (0,1)$, solving for $\widehat{y}(p_I)$ and then plotting $p_I \to \phi_\iota(p_I)$. However, we offer an analytic proof in the following proposition.

Proposition 4.2. *For fixed $\kappa > 0$ and $\lambda \in (0,1)$ the map ϕ_ι defined in (35) is strictly increasing in p_I. Therefore, $CE^I_{0-,\iota}$ is strictly increasing in the precision p_I.*

On the other hand, for the price-taking equilibrium, we have the map

$$p_I \to \phi(p_I) := \frac{(1-\lambda)^2\kappa p_I + (1+p_I)(\lambda + \kappa p_I)^2}{\kappa(1+\lambda p_I + \kappa p_I(1+p_I))^2}.$$

As $\phi(0) = \lambda^2/\kappa$ and $\phi(\infty) = 0$, this map is clearly not increasing. In fact, it is not monotonic because

$$\phi'(0) = 1 - \lambda^2 + \frac{\lambda^2}{\kappa}(1-2\lambda).$$

When $\lambda \leq 1/2$ (equivalently $\alpha_I \leq \alpha_U$ or that insider is less risk tolerant than uniformed traders), ϕ is increasing at 0. However, when the insider is relatively more risk tolerant ($\lambda > 1/2$), ϕ will be decreasing at 0 for $\kappa = \alpha_I^2 p_N$ small enough (which can happen if the noise trader variance/volume is very large).

Interestingly enough, there are cases where the certainty equivalents with and without price impact *have the opposite monotonicity with respect to signal's precision.* This is pictured in Figure 1 which shows when insider does not internalize her price impact the better quality of her signal, the lower her ex-ante expected utility becomes; while under price impact we have the more reasonable situation where price impact materializes the better quality of the signal to higher insider's certainty equivalent.

Remark 4.3. The above discussion highlights a very interesting feature of the price-impact model. If the insider does not internalize her price impact, her welfare created by the private signal is not necessarily increasing in the signal's quality. This means that in the PT equilibrium, it is not always worth it to obtain a better signal, as it may decrease utility. On the other hand, when she does internalize her price impact (and the other traders are price-takers), it is always beneficial for the insider to try and improve the quality of her private signal, as long as this improvement comes with marginally lower cost than the corresponding increase of certainty equivalent[12].

[12]Note that beneficial price-impact for the insider does not mean that the uniformed traders suffer loss of utility because of price impact. On the contrary, when uniformed traders are the same side of trade with the insider, not

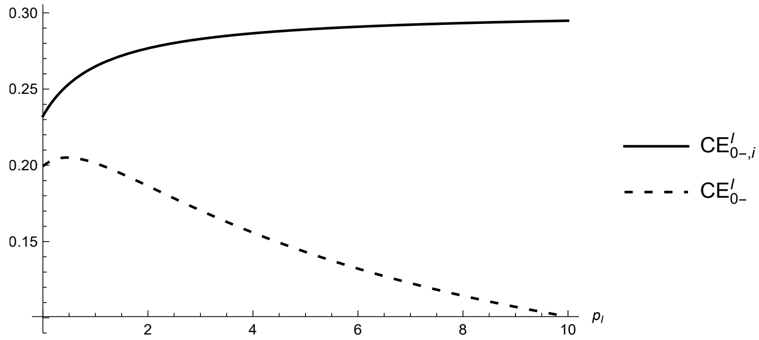

FIGURE 1. Plot comparing $\mathrm{CE}^I_{0-,\iota}$ and CE^I_{0-} as a function of p_I. Parameters are $\alpha_I = \alpha_U = .3, \mu_X = .5, P_X = 1, p_N = 1, \Pi = 0$.

Price-impact and price-taking welfare comparison. Next, we focus on how price impact internalization affects both informed and uniformed trader's welfare. In particular, we examine whether the internalization of price impact implies higher welfare for the insider and the uniformed trader, when compared to the price taking case. Using Proposition 4.1 we readily get the relations

$$\begin{aligned} &\mathrm{CE}^I_{0-,\iota} \geq \mathrm{CE}^I_{0-} \iff \frac{\kappa p_I(1+p_I)+\widehat{y}^2}{(1+p_I)(1+2\widehat{y})} \geq \frac{(1-\lambda)^2\kappa p + (1+p_I)(\lambda+\kappa p_I)^2}{(1+\lambda p_I + \kappa p_I(1+p_I))^2}; \\ &\mathrm{CE}^U_{0-,\iota} \geq \mathrm{CE}^U_{0-} \iff \frac{\kappa p_I + (1+\widehat{y})^2}{(1-\lambda)^2(1+2\widehat{y})^2} \geq \frac{1+\kappa p_I}{(1+\lambda p_I + \kappa p_I(1+p_I))^2}. \end{aligned} \tag{36}$$

Our first result shows that insider welfare need not increase when she internalizes her price impact.

Proposition 4.4. *Under Assumptions 1.1 and 2.6, both $CE^I_{0-,\iota} > CE^I_{0-}$ and $CE^I_{0-,\iota} < CE^I_{0-}$ are possible.*

Proof of Proposition 4.4. Numerically, this is demonstrated in Figure 2. Indeed, in the joint combination of high α_U (uninformed close to risk neutrality) and low-to-moderate $p_I \in (0,2)$ (modest signal quality) welfare may decrease. Analytically, this will be shown in Proposition 4.6. □

While no uniform statement can be made for the insider, our next result shows for the uninformed trader, at the ex-ante level the insider's internalization of price impact is always beneficial, at least provided the traders' initial allocation is Pareto.

Proposition 4.5. *Under Assumptions 1.1 and 2.6, $CE^U_{0-,\iota} \geq CE^U_{0-}$.*

As already mentioned, we give the economic intuition behind these model's predictions in Section 5. For now, we state a couple of additional results which will clarify more the situation.

Risk tolerance asymptotics. It is rather complicated to precisely describe the set of input parameters that characterizes the order of insider's certainty equivalents $\{(\kappa, p_I, \lambda) \mid \mathrm{CE}^I_{0-,\iota} \geq \mathrm{CE}^I_{0-}\}$. From Figure 2, we conjugate however that the situation becomes clearer when we consider the asymptotics with respect to traders' risk tolerances. For this, in the following proposition, we take limits as the insider and uninformed traders' risk tolerance go to 0 and ∞.

Proposition 4.6. *Let Assumptions 1.1 and 2.6 hold and assume p_I, p_N are fixed. Then,*

(1) Fix α_I. As $\alpha_U \to 0$, CE^I_{0-} remains bounded while $CE^I_{0-,\iota} \to \infty$. As $\alpha_U \to \infty$,

$$\lim_{\alpha_U \to \infty} \left(CE^I_{0-,\iota} - CE^I_{0-} \right) > 0.$$

only they gain welfare due to price impact, but their gain may be even higher than the insider's (e.g. under absence of private signal, see Proposition 4.7 and Remark 4.8 below).

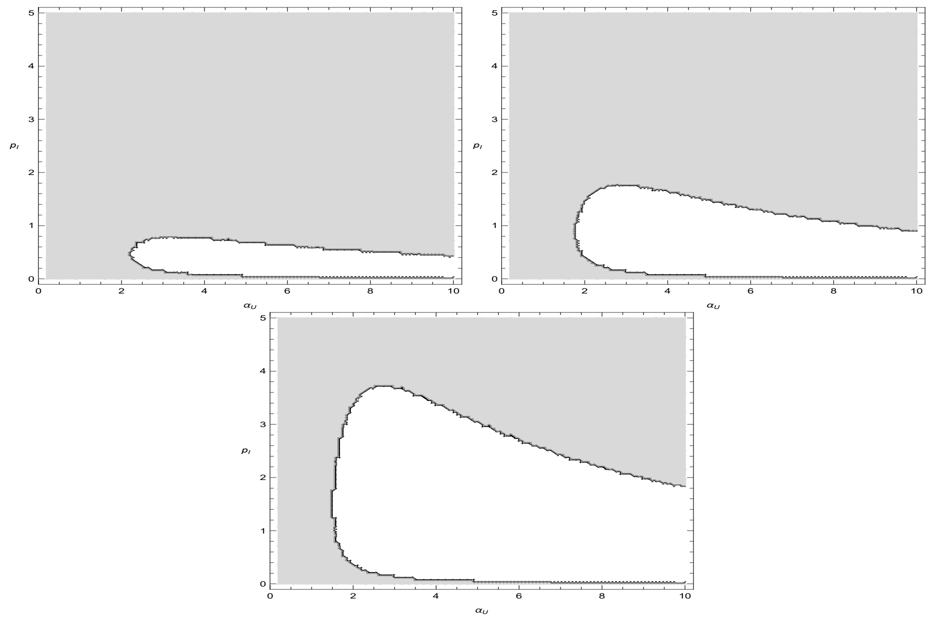

FIGURE 2. Plot comparing $\mathrm{CE}^I_{0-,\iota}$ and CE^I_{0-} as a function of α_U (x-axis) and p_I (y-axis) for $p_N = 1$ and $\alpha_I = .2$ (upper left), $\alpha_I = .1$ (upper right), $\alpha_I = .05$ (lower). The shaded region is where $\mathrm{CE}^I_{0-,\iota} > \mathrm{CE}^I_{0-}$. The white region is where $\mathrm{CE}^I_{0-,\iota} < \mathrm{CE}^I_{0-}$

(2) Fix α_U. As $\alpha_I \to 0$, $\lim_{\alpha_I \to 0}(CE^I_{0-,\iota} - CE^I_{0-})/\alpha_I = 0$ but

$$\lim_{\alpha_I \to 0} \frac{CE^I_{0-,\iota} - CE^I_{0-}}{\alpha_I^3} = \frac{d}{2} \frac{(1+p_I)^2(1+p_I-\alpha_U^2 p_I p_N)}{\alpha_U^2(\alpha_U^2 p_I p_N + 1 + p_I)}.$$

Thus, for α_I small, $CE^I_{0-,\iota} \geq CE^I_{0-}$ if and only if $1 + p_I - \alpha_U^2 p_I p_N \geq 0$. As $\alpha_I \to \infty$,

$$\lim_{\alpha_I \to \infty} \left(CE^I_{0-,\iota} - CE^I_{0-}\right) > 0.$$

(3) If $\alpha_I = \alpha_U$, then $CE^I_{0-,\iota} \geq CE^I_{0-}$.

Proposition 4.6 implies the risk aversion is a crucial parameter for the certainty equivalents' comparisons. Indeed, when traders have the same risk aversion internalizing price impact always increases agents' welfare. On the other hand, fixing α_U, p_N, insider welfare may be lower than in the price-taking case when she is very risk averse and the following structure condition holds

$$\alpha_U^2 p_N > \frac{1}{p_I} + 1. \tag{37}$$

For example, this condition holds if the uninformed trader is sufficiently risk tolerant, or/and the noise traders' demand (approximated by its variance) is sufficiently low.

Welfare in the absence of private information. We conclude our welfare analysis proving that if one turns off the asymmetric information channel, and focuses solely on effects due to internalizing of price impact, then remarkably, insider welfare increases at the interim level. As mentioned above, the situation with no private signal assumes only the insider possesses and exploits her price impact. The other (non-noise) traders act as price takers. This corresponds to when a (large) risk averse agent acts strategically, even when she does not have access to a private signal, and when the uninformed traders represent the mass of all other rational agents who are relatively small (i.e.

possess no market power) and who also have no private signal. In sum, we are measuring the effects of assuming heterogeneity, not in information, but in internalization of price impact.

For this, recall from Proposition 2.12 that the no-signal equilibrium corresponds to taking $p_I \to 0$. Using the interim welfare formulas in Lemmas C.1 and C.2 we obtain the associated no-signal equilibrium quantities in the following proposition, proved in Appendix C. To state it, recall from Section 1 that

$$X = \mu_X + P_X^{-1/2}\mathcal{E}_X; \quad G = X + \frac{1}{\sqrt{p_I}}P_X^{-1/2}\mathcal{E}_I; \qquad Z_N = \frac{1}{\sqrt{p_N}}P_X^{1/2}\mathcal{E}_N, \tag{38}$$

where $\mathcal{E}_X, \mathcal{E}_I, \mathcal{E}_N$ are three independent $N(0, 1_d)$ random variables. Lastly, recall λ from (28).

Proposition 4.7. *Let Assumption 2.6 hold. As $p_I \to 0$ we obtain the almost sure limits for the insider*

$$\lim_{p_I\to 0}\frac{1}{\alpha_I}CE_0^I(G,Z_N) = \psi_{I,0}'\mu_X - \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0} + \frac{1}{2}\left\| P_X^{-1/2}\left(\frac{\lambda}{\alpha_I}Z_N + (1-\lambda)(\psi_{I,0}-\psi_{U,0})\right)\right\|^2,$$

$$\begin{aligned}\lim_{p_I\to 0}\frac{1}{\alpha_I}CE_{0,\iota}^I(G,Z_N) &= \psi_{I,0}'\mu_X - \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0}\\ &\quad + \frac{1}{2(1-\lambda^2)}\left\| P_X^{-1/2}\left(\frac{\lambda}{\alpha_I}Z_N + (1-\lambda)(\psi_{I,0}-\psi_{U,0})\right)\right\|^2.\end{aligned}$$

Therefore, insider interim welfare always increases when internalizing price impact. For the uninformed agent we obtain almost surely

$$\lim_{p_I\to 0}\frac{1}{\alpha_U}CE_0^U(H) = \psi_{U,0}'\mu_X - \frac{1}{2}\psi_{U,0}'P_X^{-1}\psi_{U,0} + \frac{1}{2}\left\| P_X^{-1/2}\left(\frac{\lambda}{\alpha_I}Z_N + \lambda(\psi_{U,0}-\psi_{I,0})\right)\right\|^2,$$

$$\begin{aligned}\lim_{p_I\to 0}\frac{1}{\alpha_I}CE_{0,\iota}^U(H_\iota) &= \psi_{U,0}'\mu_X - \frac{1}{2}\psi_{U,0}'P_X^{-1}\psi_{U,0}\\ &\quad + \frac{1}{2(1-\lambda)^2(1+\lambda)^2}\left\| P_X^{-1/2}\left(\frac{\lambda}{\alpha_I}Z_N + (1-\lambda)\lambda(\psi_{U,0}-\psi_{I,0})\right)\right\|^2,\end{aligned}$$

and hence interim welfare may be higher or lower in the price impact case.

Remark 4.8. Two quite surprising consequences stem from the above proposition. First, that the insider's welfare might be higher in the PT equilibrium is directly attributable to the presence of the private signal, as when there is no private signal, internalization of price impact is always beneficial for the insider. Second, assuming no private signal and the initial Pareto allocation of Assumption 1.1, we have the following almost sure order of interim certainty equivalents

$$\mathrm{CE}_{0,\iota}^U > \mathrm{CE}_{0,\iota}^I > \mathrm{CE}_0^U = \mathrm{CE}_0^I.$$

Amazingly, not only it is better for the uninformed agent when the insider internalizes her price impact, the uninformed agent's welfare actually exceeds that of the insider. We explain the mechanism behind these predictions in the next section.

5. Equilibria Structure

In this section, we analyze and compare the equilibrium quantities (allocation and prices) in order to infer the economic intuition behind the models predictions on the effects of price-impact internalization ("internalization") and insider's private signal ("private information"). As will be shown, internalization and private information may have competing affects on the insider's demand, and hence equilibrium quantities. Broadly, internalization has the effect of dampening the position size, while private information may increase the insider's equilibrium allocation of risk.

We will compare when the insider (1) takes prices as given versus internalizing price impact, and (2) when the insider has a private signal versus when there is no private signal. This leads to four cases. The first two sections deal with the effect of information asymmetry, starting with the price taking (PT) equilibrium and then moving on to the price impact (PI) equilibrium. The last two sections deal with the effect of price impact internalization, staring when there is no private information, and then considering the private signal case. In each section we compare the equilibrium structure of demands and prices. Throughout we use the notation in (28), and whenever we consider the PI equilibria, we will enforce Assumption 2.6. Although all the formulas are provided in Appendices without restrictions on d, Π and initial endowments, in order to simplify the discussion of this section we will also impose Assumption 1.1 and consider a single tradeable asset (i.e X is of dimension one), which outstanding supply Π is positive.

Private information effects when price taking. We start with the PT equilibrium, identifying the effect of the signal on the demands, as well as the price. Using (15), (21), (23), (24) and (27) the insider's optimal demand functions satisfy

$$\begin{aligned}\widehat{\psi}_I(g,z) - \widehat{\psi}_{ns,I}(z) &= (P_I - P)(P+P_X)^{-1}P_X\left(g - \mu_X + P_X^{-1}\widehat{\Pi}\right) \\ &\quad - \left((P_I + P_X)(P+P_X)^{-1}PP_I^{-1} - \lambda 1_d\right)\frac{z}{\alpha_I}.\end{aligned} \tag{39}$$

From (23) we see that $P_I - P$ is positive definite, and thus (as expected) insider optimal demand relative to the no-signal case is increasing in the signal. Also, as the matrix in front of z is negative definite, we see that insider relative demand is decreasing in the noise trader demand.

Another interesting effect concerns the outstanding supply, $\Pi = (\alpha_I + \alpha_U)\widehat{\Pi}$. When the supply is positive, the presence of a private signal (no matter the value) increases the insider's demand. Furthermore, at the ex-ante level, the expected demand change is

$$\mathbb{E}\left[\widehat{\psi}_I(G, Z_N) - \widehat{\psi}_{ns,I}(Z_N)\right] = (P_I - P)(P+P_X)^{-1}\widehat{\Pi}, \tag{40}$$

so that a positive supply will increase the insider's position on average. Intuitively, positive supply means that the insider and the uniformed trader are expected to buy the tradeable asset at the equilibrium and for the insider the presence of private signal means that her estimated variance of the tradeable asset (i.e. its risk) is lower. This increases her demand, implying that ex-ante she is more confident to hold higher part of asset supply.

The private signal's affect on insider demand is transferred to the equilibrium price, as an increase in the insider's demand tends to increase the price as well. Indeed, from (23) and (27) we find

$$p(h(g,z)) - p_{ns}(z) = (P_X + P)^{-1}\left(P\left(g - \mu_X + P_X^{-1}\widehat{\Pi}\right) + \left(PP_I^{-1} - \lambda(P+P_X)P_X^{-1}\right)\frac{z}{\alpha_I}\right). \tag{41}$$

This shows the equilibrium price is increasing in the signal. In terms of ex-ante expectation, the sign of $\mathbb{E}\left[p(h(G,Z_N)) - p_{ns}(Z_N)\right] = (P_X + P)^{-1}PP_X^{-1}\widehat{\Pi}$ coincides with that of $\widehat{\Pi}$. Thus, positive $\widehat{\Pi}$ implies the presence of the signal increases both the insider's expected demand and expected equilibrium price, with the opposite conclusion when $\widehat{\Pi}$ is negative. In addition, we get from the equilibrium clearing condition (2) that the private signal has the opposite effect on the uniformed trader's demand[13]. Summing up, we state

> *For positive outstanding supplies, the presence of private information in the PT equilibria is expected to increase the insider's demand and the price and to decrease the uniformed trader's demand.*

[13]The uninformed agent sees Z_N (through the price) in the no-signal equilibrium, but does not see Z_N in the PT equilibrium. Therefore, we are not saying the uninformed agent sees Z_N in the PT equilibrium and then adjusts her position accordingly. Rather, we are saying the effect of noise trading in the price taking equilibrium is to increase the trade size of the uninformed agent over the no-signal equilibrium.

Private information effects when internalizing price impact. The effects of private signal on equilibrium prices and demands in the PI equilibrium are similar as in the PT equilibrium. Using Proposition 2.12 and (64) below, we decompose the insider's demand under internalization and signal as follows

$$\widehat{\psi}_{I,\iota}(g,z)-\widehat{\psi}_{ns,\iota,I}(z)=\frac{p_I}{1+2\widehat{y}}P_X\left(g-\mu_X+P_X^{-1}\widehat{\Pi}\right)-\frac{(1-\lambda)\widehat{y}-\lambda}{(1+2\widehat{y})(1+\lambda)}\frac{z}{\alpha_I}.$$

As expected, the insider demand (relative to the no-signal case) is increasing with the private signal. Similarly, by using the arguments in the proof of Proposition 3.4, one can show $\widehat{y}>\lambda/(1-\lambda)$ and hence the coefficient in front of z/α_I is negative.

As in the PT equilibrium, at the ex-ante level, when outstanding supply is positive, the presence of signal increases the insider's demand because

$$\mathbb{E}\left[\widehat{\psi}_{I,\iota}(G,Z_N)-\widehat{\psi}_{ns,\iota,I}(Z_N)\right]=\frac{p_I}{1+2\widehat{y}}\widehat{\Pi}. \tag{42}$$

As for the equilibrium prices, using (15), (31) and (32), we obtain

$$p_\iota(h_\iota(g,z))-p_{ns,\iota}(z)=\frac{p_I\widehat{y}}{(1+p_I)(1+2\widehat{y})}\left(g-\mu_X+P_X^{-1}\widehat{\Pi}\right)+\left(\frac{\widehat{y}(1+\widehat{y})}{(1+p_I)(1+2\widehat{y})}-\frac{\lambda}{1-\lambda^2}\right)\frac{z}{\alpha_I}.$$

Therefore, the relative price change is increasing in the signal and (positive) outstanding supply (this is due to the corresponding increased insider demand). At the ex-ante level, we have

$$\mathbb{E}\left[p_\iota(h_\iota(G,Z_N))-p_{ns,\iota}(Z_N)\right]=\frac{p_I\widehat{y}}{(1+p_I)(1+2\widehat{y})}P_X^{-1}\widehat{\Pi},$$

which shows a positive difference for positive supplies. This is the same as in the PT case, and in fact, using (41), along with the notation of (28), (32) and (33) we obtain

$$\mathbb{E}\left[p(h(G,Z_N))-p_{ns}(Z_N)\right]-\mathbb{E}\left[p_\iota(h_\iota(G,Z_N))-p_{ns,\iota}(Z_N)\right]=(m_g-m_{g,\iota})P_X^{-1}\widehat{\Pi}.$$

In view of Proposition 3.4, this means that the expected price change caused by the presence of private signal is lower when accounting for price impact (consistent with the reduced sensitivity with respect to public signal that internalization yields).

As in the PT equilibrium, by market clearing, the effects of signal to the uniformed trader's demand is of opposite direction. For this, we directly calculate that

$$\mathbb{E}\left[\widehat{\psi}_{U,\iota}(Z_N)-\widehat{\psi}_{ns,\iota,U}(Z_N)\right]=-\frac{p_I\lambda}{(1-\lambda)(1+2\widehat{y})}\widehat{\Pi},$$

which means that in contrast to the insider, a positive outstanding supply is expected to decrease the uniformed trader's demand due to price impact. As a bottom line of the above analysis we may conclude

> *For both the PT and PI equilibria, for positive outstanding supplies, the presence of a private signal is expected to increase the insider's demand and prices (albeit with a lower change in the PI equilibrium) and decrease the uniformed trader's demand.*

Price impact internalization effects when there is no information asymmetry. We now turn our attention to the effect of price impact, first assuming absence of private information. According to Proposition 2.12, we first have the equilibrium allocation

$$\widehat{\psi}_{ns,I}(z)-\psi_{I,0}=-\lambda\frac{z}{\alpha_I};\qquad \widehat{\psi}_{ns,\iota,I}(z)-\psi_{I,0}=-\frac{\lambda}{1+\lambda}\frac{z}{\alpha_I},$$

which gives

$$\widehat{\psi}_{ns,\iota,I}(z)-\widehat{\psi}_{ns,I}(z)=\frac{\lambda^2}{(1+\lambda)}\frac{z}{\alpha_I}.$$

Therefore, internalization of price impact keeps the insider at the same side of the trade as in the non-internalization case, but it *reduces* the magnitude of the trade. Intuitively, the insider accounts for price impact by taking a smaller position, which in turn changes the equilibrium price. Indeed, we readily get that

$$p_{ns,\iota}(z) - p_{ns}(z) = \frac{\lambda^3}{1-\lambda^2} P_X^{-1} \frac{z}{\alpha_I}.$$

The above implies that in the PI equilibrium, the insider has a lower demand when compared to the PT equilibrium and obtains a better price (price-impact increases the per-unit price when insider sells and decreases it when she buys). Indeed, under Assumption 1.1, positive z means that both insider and uniformed trader sell at equilibrium. In the view of Remark 3.2, positive z makes the insider reveal a higher public signal. This increases the demand of the uniformed trader (covers less of noise demand) and hence the interim effect is an increase in the equilibrium price. Lastly, the uniformed trader's equilibrium allocations are

$$\widehat{\psi}_{ns,U}(z) - \psi_{U,0} = -\lambda \frac{z}{\alpha_I}; \qquad \widehat{\psi}_{ns,\iota,U}(z) - \psi_{U,0} = \frac{\lambda}{1-\lambda^2} \frac{z}{\alpha_I},$$

as well as relative trade size

$$\widehat{\psi}_{ns,\iota,U}(z) - \widehat{\psi}_{ns}(z) = -\frac{\lambda^3}{1-\lambda^2} \frac{z}{\alpha_I}.$$

Therefore, when $z < 0$ both the insider and uninformed trader buy the risky asset in each equilibria. However, in the PI equilibrium, the insider reduces her position while the uninformed trader increases it. Conversely, when $z > 0$ traders sell the risky asset, with the uniformed trader selling more. In particular, the uninformed increases his volume at a better price-per-unit, due to price impact. In other words, when insider buys the asset, internalization reduces her demand which in turn decreases the price and makes the uniformed trader buys more. Summarizing,

> *Internalization with no signal results in a lower (resp. higher) equilibrium position for the insider (resp. uniformed trader) at a better price.*

Price impact internalization effects when there is information asymmetry. We finally consider the effect of price-impact internalization on equilibrium demands and prices in the presence of private information, which associates with our main case. To simplify the presentation and highlight the key points, we state the results at the expected value level, rather than for each realization.

We have seen that the presence of signal is expected to increase the volume of the insider's order, while the internalization is expected to decrease it (under no private signal). In other words, price impact and presence of the signal have ex-ante opposite expected effects on the insider's demand. In particular, using Proposition 2.12, (28), (40) and (42) we obtain

$$\mathbb{E}\left[\widehat{\psi}_{I,\iota}(G, Z_N) - \widehat{\psi}_I(G, Z_N)\right] = p_I \left(\frac{1}{1+2\widehat{y}} - \frac{1-\lambda}{(1+p_I)(\lambda+\kappa p_I)+(1-\lambda)} \right) \widehat{\Pi} \tag{43}$$

The following result shows that the right side of (43) may be either positive or negative. In short, the effect of price impact prevails over the one of asymmetric information when the insider is sufficiently risk tolerant resulting is a higher expected order.

Proposition 5.1. *Let Assumptions 1.1 and 2.6 hold. Then, internalization of price-impact is ex-ante expected to increase (resp. decrease) the size order when the insider is sufficiently risk tolerant (resp. risk averse). As such, by market clearing, the internalization of price-impact induces the opposite ex-ante expected effect for the uniformed trader.*

Remark 5.2. The conclusions of Proposition 5.1 when the insider is risk tolerant also hold when the uninformed trader is sufficiently risk averse in that $\alpha_U \to 0$, provided $\kappa p_I(1+p_I) > 2$, which corresponds to $\alpha_I^2 p_I p_N(1+p_I) > 2$ (e.g. the private signal is of good quality). Indeed, we have that

$\lambda \to 0$ so that, in the proof of Proposition 4.6, $\overline{\kappa} \to 2/(p_I(1+p_I))$. We also see that if the insider's signal is sufficiently precise (i.e. $p_I \to \infty$) then $\overline{\kappa} \to 0$ while $\kappa = \alpha_I^2 p_N$ and hence the insider is expected to increase the size of her order (over the price-taking case).

Lastly, for the equilibrium prices, (32) and (33) yield

$$\mathbb{E}\left[p_\iota(H_\iota) - p(H)\right] = (m_{g,\iota} - m_g) P_X^{-1} \widehat{\Pi} \tag{44}$$

Thanks to Proposition 3.4, we see that the factor in front of $\widehat{\Pi}$ is negative, which means that internalization is expected to decrease (resp. increase) the prices when insider is expected to buy (resp. sell). In other words, the expected change of the price benefits both traders (as the uniformed trader remains at the same side of trade). Connecting this fact to the expected demand changes, we may conclude that

> *Due to internalization of price impact, a sufficiently low (resp. high) risk tolerant insider is expected to buy less (resp. more) units at a better price, while uniformed trader buys more (resp. less).*

Intuition and model's predictions. We conclude by providing economic intuition about the predictions induced by the model. To focus the discussion on price impact and asymmetric information, we consider again the case of Pareto initial allocation (Assumption 1.1) a single asset and the reasonable situation of positive outstanding supply.

We have seen that asymmetric information is ex-ante expected to increase the insider's demand for the tradeable assets. Intuitively, the private signal reduces the asset's risk (measured by variance) for the insider, which makes her willing to hold a higher allocation. Without strategic trading, higher insider's demand is expected to increase prices. Private information tends to increase the insider's demand even under price impact. The main ex-ante expected difference is the lower increase of prices, as internalization of price impact affects prices in favor of the insider. When the insider trades strategically, she uses her private signal to affect the equilibrium. In fact, it is exactly when the insider internalizes price impact and uninformed agents are price-takers that insider's welfare is monotonically increasing with respect to the signal precision. In other words, insider's strategic trading increases the value of her private signal making the acquisition of better signal reasonable.

Internalization of price impact keeps the traders at the same side of trade. As we have seen, the insider hides part of her private signal, which is expected to reduce her demand and hence the prices. In other words, price impact has the opposite expected effect than the private signal. Provided that insider has long position at equilibrium, prices always decrease due to price impact, under both price impact and private signal. However, the insider's demand is lower under the presence of signal when she is also sufficiently risk averse. This is intuitive in the sense that a sufficiently risk averse trader wants to undertake less risk. If in addition to high insider's risk aversion inequality (37) holds, the price impact equilibrium lowers the insider's welfare. Note that (37) implies the uniformed traders are highly risk tolerant. This reduces the insider's allocation even further, since the uniformed traders' demand is higher and hence at market-clearing the insider's share is lower. Such lower insider's demand may lead to lower ex-ante expected utility gains, as a signal of good quality (consistent with (37)) means the insider feels more confident to hold a higher position at equilibrium. However, under a large deviation of risk aversions, internalization of price impact prevails, demand is reduced and utility gains are lower. Note that when traders have the same risk aversion, the effect of internalization is always beneficial for the insider.

Interestingly enough, in the absence of a private signal, internalization of price impact induces higher utility gains for the insider. This is because under symmetric information, the insider does not have motive to increase her demand due to a lower asset risk, and hence the effect of price impact, i.e. buying lower quantity at a lower price, increases the expected welfare. We conclude that it is the presence of asymmetric information and the deviation on risk aversions that potentially make the price-impact equilibrium disadvantageous for the insider.

We should also emphasize that as long as the insider internalizes her price impact, equilibrium prices cannot be driven to the corresponding PT equilibrium. This is because the uniformed trader, although he is a price-taker, realizes the insider internalizes her price impact. This makes him reduce his perceived public signal precision, and hence alters his demand function to a less elastic one. Under the Pareto initial allocation, he remains at the same side of trade with the insider and the lower equilibrium prices caused by the internalization imply higher demand for the uniformed trader, who (although price-taker) is benefited by price impact. In other words, price impact decreases the prices when traders buy the asset, and at equilibrium the uniformed trader satisfies his optimal demand but at a discount. This is the reason why price impact ex-ante benefits the uniformed trader with and without asymmetric information.

Considering the aggregate utility gains of both the insider and uniformed traders at both equilibrium, we may conclude that as long as (37) does not hold, the internalization of price impact increases the non-noise traders' welfare with and without the presence of private signal.

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Appendix A. Proofs from Section 2

Proof of Lemma 2.1. The law of X given $\sigma(G, Z_N)$ has density

$$\frac{\mathbb{P}\left[X\in dx\middle|\sigma(G,Z_N)\right]}{\mathbb{P}\left[X\in dx\right]} = \frac{\mathbb{P}\left[X\in dx\middle|\sigma(G)\right]}{\mathbb{P}\left[X\in dx\right]} = \left(\frac{e^{-\frac{1}{2}x'P_Ix+x'P_Ig}}{\mathbb{E}\left[e^{-\frac{1}{2}X'P_IX+X'P_Ig}\right]}\right)\Bigg|_{g=G}. \tag{45}$$

Using this in (6), on $\{G=g, Z_N=z\}$ the insider minimizes over $\psi=\psi(g,z)$ with $\psi(G,Z_N)\in\mathcal{A}_I$

$$(\psi-\psi_{I,0})'p_\iota\left(\psi,z\right)+\log\left(\frac{\mathbb{E}\left[e^{-\psi'X-\frac{1}{2}X'P_Ix+X'P_Ig}\right]}{\mathbb{E}\left[e^{-\frac{1}{2}X'P_IX+X'P_Ig}\right]}\right).$$

As $X\sim N(\mu_X, P_X^{-1})$, this specifies to

$$(\psi-\psi_{I,0})'p_\iota\left(\psi,z\right)+\frac{1}{2}\psi'(P_I+P_X)^{-1}\psi-\psi'\left(P_I+P_X\right)^{-1}\left(P_Ig+P_X\mu_X\right).$$

Plugging in for p_ι, using $\psi'M\psi=(1/2)\psi'(M+M')\psi$, and grouping by powers of ψ gives

$$\begin{aligned}&\frac{1}{2}\psi_{I,0}'M\left(\psi_{I,0}-\frac{z}{\alpha_I}\right)-\psi_{I,0}'V+\frac{1}{2}\psi'\left(M+M'+(P_I+P_X)^{-1}\right)\psi\\&\quad-\psi'(P_I+P_X)^{-1}\left(P_Ig+P_X\mu_X+(P_I+P_X)\left((M+M')\psi_{I,0}-M\frac{z}{\alpha_I}-V\right)\right).\end{aligned}\tag{46}$$

Plugging in for M and using $P_{X|G}$ from (7), the optimizer $\widehat{\psi}_I$ is

$$\begin{aligned}\widehat{\psi}_{I,\iota}(g,z) = P_{X|G}^{1/2}(\mathcal{Y}+\mathcal{Y}'+1_d)^{-1}P_{X|G}^{-1/2}\Big(& P_I g + P_X\mu_X - P_{X|G}V + P_{X|G}^{1/2}(\mathcal{Y}+\mathcal{Y}')P_{X|G}^{-1/2}\psi_{I,0} \\ & - P_{X|G}^{1/2}\mathcal{Y}P_{X|G}^{-1/2}\frac{z}{\alpha_I}\Big).\end{aligned} \tag{47}$$

The identity in (8) with $\mathcal{M},\Lambda_\iota,\mathcal{V}$ from (9) follow by direct computations □

Proof of Lemma 2.3. As the uninformed is a price taker, this follows immediately from Proposition 2.9 below, with the appropriate substitutions $p\to p_\iota$, $H\to H_\iota$. □

Proof of Proposition 2.4. Throughout we suppress the dependence of all quantities on $\mathcal{Y}$. Now, (8), (12) and (13) show the right side of (14) is affine in the signal H_ι, and after dividing by α_U and right-multiplying by $\mathcal{M}^{-1}$, (16) is precisely the equation which eliminates the H_ι terms. As for the constant terms, we need

$$\Pi = (\alpha_I 1_d - \alpha_U(P_{U,\iota}+P_X)M)\,\mathcal{M}\mathcal{V} + \alpha_I\psi_{I,0} + \alpha_U\left(P_X\mu_X - (P_{U,\iota}+P_X)V\right).$$

As (16) does not involve V, if $\widehat{\mathcal{Y}}$ is such that (16) holds, the above becomes

$$\begin{aligned}\Pi &= -\alpha_U P_{U,\iota}\mathcal{V} + \alpha_I\psi_{I,0} + \alpha_U\left(P_X\mu_X - (P_{U,\iota}+P_X)V\right),\\ &= -\alpha_U P_I^{-1}P_{U,\iota}\left(P_X\mu_X - \psi_{I,0} - P_{X|G}V\right) + \alpha_I\psi_{I,0} + \alpha_U\left(P_X\mu_X - (P_{U,\iota}+P_X)V\right),\\ &= \alpha_U(P_I - P_{U,\iota})P_I^{-1}P_X(\mu_X - V) + (\alpha_I P_I + \alpha_U P_{U,\iota})\psi_{I,0},\end{aligned}$$

where the last equality follows from (7), (9) and simplifying. Therefore, using $\widehat{\Pi}$ and p_0 from (4), and P_ι from (17) we deduce

$$\begin{aligned}\widehat{\Pi} &= \frac{\alpha_U}{\alpha_I+\alpha_U}\left(P_I - P_{U,\iota}\right)P_I^{-1}P_X\left(\mu_X - V\right) + \frac{\alpha_U P_{U,\iota}+\alpha_I P_I}{\alpha_I+\alpha_U}P_I^{-1}\psi_{I,0},\\ &= \left(P_I - P_\iota\right)P_I^{-1}P_X\left(p_0 - V\right) + \left(P_I - P_\iota\right)P_I^{-1}\widehat{\Pi} + P_\iota P_I^{-1}\psi_{I,0}.\end{aligned}$$

This yields

$$V = p_0 + P_X^{-1}P_I\left(P_I - P_\iota\right)^{-1}P_\iota P_I^{-1}\left(\psi_{I,0} - \widehat{\Pi}\right).$$

(17) now follows as $P_I(P_I-P_\iota)^{-1}P_\iota P_I^{-1} = P_\iota(P_I-P_\iota)^{-1}$. As for (18), we have from (12) that

$$p_\iota(h_\iota) = p_0 + M\mathcal{M}(h_\iota - p_0) + (M\mathcal{M} - 1_d)\,p_0 + M\mathcal{M}\mathcal{V} + V.$$

Next,

$$\begin{aligned}&(M\mathcal{M} - 1_d)\,p_0 + M\mathcal{M}\mathcal{V} + V\\ &\quad= (M\mathcal{M} - 1_d)\,p_0 + V + M\mathcal{M}P_I^{-1}\left(P_X p_0 + \widehat{\Pi} - \psi_{I,0} - P_{X|G}V\right),\\ &\quad= (M\mathcal{M} - 1_d)\,p_0 + p_0 + P_X^{-1}P_\iota(P_I - P_\iota)^{-1}\left(\psi_{I,0} - \widehat{\Pi}\right)\\ &\qquad\quad + M\mathcal{M}P_I^{-1}\left(P_X p_0 - \left(\psi_{I,0} - \widehat{\Pi}\right) - P_{X|G}p_0 - P_{X|G}P_X^{-1}P_\iota(P_I - P_\iota)^{-1}\left(\psi_{I,0} - \widehat{\Pi}\right)\right).\end{aligned}$$

Because $P_{X|G} = P_I + P_X$ the p_0 terms cancel out. The $\psi_{I,0} - \widehat{\Pi}$ terms are

$$\begin{aligned}&-M\mathcal{M}P_I^{-1} + \left(1_d - M\mathcal{M}P_I^{-1}(P_X + P_I)\right)P_X^{-1}P_\iota(P_I - P_\iota)^{-1}\\ &\quad= \left(P_X^{-1} - M\mathcal{M}(P_\iota^{-1} + P_X^{-1})\right)P_\iota(P_I - P_\iota)^{-1}.\end{aligned}$$

□

Proof of Proposition 2.7. Using (30), and assuming $\mathcal{Y} = y\ 1_d$, (16) becomes $0_d = f(y)1_d$ where

$$f(y) = \frac{\alpha_I}{\alpha_U((1+y)^2) + \alpha_I^2 p_N p_I}\left((1+y)^2\left(1 - \frac{\alpha_U y}{\alpha_I(1+p_I)}\right) + \alpha_I^2 p_N p_I\left(\frac{\alpha_U}{\alpha_I}(1+y)+1\right)\right).$$

This shows that we seek positive solutions to the cubic equation in (19). Define $g(y)$ as the cubic function on the right side of (19). It is clear that $g(0) > 0$ and $\lim_{y\to\infty} g(y) = -\infty$. This shows there exists a solution $\widehat{y} > 0$ to $g(\widehat{y}) = 0$. As for uniqueness of positive solutions, straight-forward computations show for any solution to $g(y) = 0$ that

$$(1+y)\dot{g}(y) = -\frac{\alpha_U}{\alpha_I}(1+y)\alpha_I^2 p_N p_I - \alpha_I^2 p_N p_I - (1+y)^3\frac{\alpha_U}{\alpha_I(1+p_I)}.$$

Thus, for any solution $y > -1$, g strictly decreasing at y and hence there is a unique solution exceeding -1, which is in fact positive.

□

Proof of Proposition 2.9. Assume the market signal takes the form $H = G + \Lambda Z_N$, for a to-be-determined matrix Λ, and the time 0 price is a function $p = p(H)$ of the market signal. Clearly, $\sigma(G,H) = \sigma(G, Z_N)$, and hence (20) is equivalent to

$$\inf_{\psi\in\sigma(G,Z_N)} \mathbb{E}\left[e^{-\psi'_{I,0}p(H) - \psi'(X - p(H))}\middle|\sigma(G,Z_N)\right].$$

We may ignore the $\psi'_{I,0}p(H)$ term, and using (45), on the set $\{G,H\} = \{g,h\}$ the insider solves

$$\inf_{\psi}\left(\psi' p(h) + \log\left(\frac{\mathbb{E}\left[e^{-\psi' X - \frac{1}{2}X'P_I X + X'P_I g}\right]}{\mathbb{E}\left[e^{-\frac{1}{2}X'P_I X + X'P_I g}\right]}\right)\right).$$

Using that $X \sim N(\mu_X, P_X^{-1})$, this problem is equivalent to minimizing

$$\frac{1}{2}\psi_I'(P_I+P_X)^{-1}\psi_I - \psi_I'(P_I+P_X)^{-1}\left(P_X\mu_X + P_I g - (P_I+P_X)p(h)\right),$$

which yields $\widehat{\psi}_I$ from (24). Next, the clearing condition (2) mandates that $\widehat{\psi}_I + Z_N/\alpha_I$ be $\sigma(H)$ measurable in equilibrium. Using (24), this in turn implies

$$\widehat{\psi}_I(G,H) + \frac{Z_N}{\alpha_I} = P_I\left(G + \frac{1}{\alpha_I}P_I^{-1}Z_N\right) + P_X\mu_X - (P_I+P_X)p(H),$$

must be $\sigma(H)$ measurable, which is achieved setting H as in (21). Turning to U, as H is of the same form as G, we may repeat the steps for the insider, but using (P_U, H) rather than (P_I, G), to conclude that $\widehat{\psi}_U$ as in (24) is optimal. We now use (2) to identify $p(H)$. Indeed, in equilibrium we must have

$$\Pi = \alpha_I\left(\widehat{\psi}_I(G,H) + \frac{Z_N}{\alpha_I}\right) + \alpha_U\widehat{\psi}_U(H),$$

which, using Assumption 1.1 and P from (23), implies

$$\begin{aligned}\widehat{\Pi} &= \frac{\alpha_I}{\alpha_I+\alpha_U}\left(P_I H + p_X\mu_X - (P_I+P_X)p(H)\right) + \frac{\alpha_U}{\alpha_I+\alpha_U}\left(P_U H + p_X\mu_X - (P_U+P_X)p(H)\right),\\ &= PH + P_X\mu_X - (P_X+P)p(H).\end{aligned}$$

Therefore, $p(H) = (P_X+P)^{-1}(PH + P_X p_0)$ which gives (23), finishing the result.

□

The proof of Proposition 2.12, as well as those in both Sections B and C simplify if we adjust the notation in (28) by defining

$$\beta := \kappa p_I = \alpha_I^2 p_I p_N; \quad \lambda_I = \lambda = \frac{\alpha_I}{\alpha_I+\alpha_U}; \quad \lambda_U = 1-\lambda = \frac{\alpha_U}{\alpha_I+\alpha_U}, \quad R := \frac{1}{1+p_I}. \tag{48}$$

The cubic equation (29) becomes

$$0=(1+y)^2\left(1-\frac{\lambda_U R}{\lambda_I}y\right)+\beta\left(\frac{\lambda_U}{\lambda_I}(1+y)+1\right). \tag{49}$$

Proof of Proposition 2.12. We use the notation in (48), and suppress $\widehat{y}$ from the $M,\mathcal{M},\mathcal{V},V$ functions. We start with the price taking equilibria. From (1), (21), (28), (32) and (48) we obtain

$$p(H)=p_0+\frac{\beta+\lambda_I}{\lambda_U R+\beta+\lambda_I}\left((1-R)X+\sqrt{R(1-R)}P_X^{-1/2}\mathcal{E}_I+\frac{1}{\alpha_I}RP_X^{-1}Z_N-(1-R)p_0\right).$$

In (48) note that $p_I\to 0$ corresponds to $R\to 1,\beta\to 0$. Therefore, almost surely

$$\lim_{p_I\to 0}p(H)=p_0+\frac{\lambda_I}{(\lambda_U+\lambda_I)}P_X^{-1}\frac{Z_N}{\alpha_I}=p_0+\lambda_I P_X^{-1}\frac{Z_N}{\alpha_I},$$

where the last equality follows because $\lambda_U+\lambda_I=1$. This gives the pricing formula in (27) as $\lambda=\lambda_I$. As for the positions, first note from (51) that

$$P_U=\frac{(1-R)\beta}{R(1+\beta)}P_X\to 0;\qquad P_U H=\frac{(1-R)\beta}{R(1+\beta)}P_X\left(X+\frac{R}{\alpha_I(1-R)}P_X^{-1}Z_N\right)\to 0,$$
$$P_I G=\frac{1-R}{R}P_X\left(X+\sqrt{\frac{R}{1-R}}P_X^{-1/2}\mathcal{E}_I\right)\to 0.$$

(27) follows from (15) and (24). We next consider the price impact case. From (48) and (49) we see that $p_I\to 0$ additionally implies $(1-R)/p_I\to 1$ and $\widehat{y}\to\lambda_I/\lambda_U$. From (1), (10), and (32)

$$\begin{aligned}p_\iota(H_\iota)=p_0&+\frac{\widehat{y}}{1+2\widehat{y}}\times\left((1-R)X+\sqrt{R(1-R)}P_X^{-1/2}\mathcal{E}_I+\frac{R(1+\widehat{y})}{\alpha_I}P_X^{-1}Z_N-(1-R)p_0\right)\\&+\frac{\lambda_I\widehat{y}\left(\beta+(1+\widehat{y})^2\right)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}\times P_X^{-1}\left(\psi_{I,0}-\widehat{\Pi}\right).\end{aligned}$$

Therefore, almost surely, and using $\lambda_U+\lambda_I=1$

$$\begin{aligned}\lim_{p_I\to 0}p_\iota(H_\iota)&=p_0+\frac{\lambda_I(\lambda_U+\lambda_I)}{(\lambda_U+2\lambda_I)\lambda_U}P_X^{-1}\frac{Z_N}{\alpha_I}+\frac{\lambda_I^2}{\lambda_U(\lambda_U+2\lambda_I)}P_X^{-1}\left(\psi_{I,0}-\widehat{\Pi}\right),\\&=p_0+\frac{\lambda_I}{1-\lambda_I^2}P_X^{-1}\frac{Z_N}{\alpha_I}+\frac{\lambda_I^2}{1-\lambda_I^2}P_X^{-1}\left(\psi_{I,0}-\widehat{\Pi}\right).\end{aligned}$$

As for I's optimal policy, from (5)

$$\widehat{\psi}(G,Z_N)=\psi_{I,0}-\frac{Z_N}{\alpha_I}+M^{-1}(p_\iota(H_\iota)-V).$$

From (30) and (48) we find that $M^{-1}\to(\lambda_U/\lambda_I)P_X$, and we have just computed the limit of $p_\iota(H_\iota)$. As for V, from (17) as well as (52) and (53) below we deduce

$$V=p_0+\frac{\beta+\lambda_I(1+\widehat{y})^2}{\lambda_U(1+\widehat{y})^2}P_X^{-1}(\psi_{I,0}-\widehat{\Pi}). \tag{50}$$

Therefore, $V \to p_0 + (\lambda_I/\lambda_U)P_X^{-1}(\psi_{I,0} - \widehat{\Pi})$. Putting everything together

$$\begin{aligned}\lim_{p_I\to 0}\widehat{\psi}_{I,\iota}(G,Z_N) &= \psi_{I,0} - \frac{Z_N}{\alpha_I} + \frac{\lambda_U}{\lambda_I}P_X\left(p_0 + \frac{\lambda_I}{1-\lambda_I^2}P_X^{-1}\frac{Z_N}{\alpha_I} + \frac{\lambda_I^2}{1-\lambda_I^2}P_X^{-1}\left(\psi_{I,0} - \widehat{\Pi}\right)\right.\\ &\qquad\left. - p_0 - \frac{\lambda_I}{\lambda_U}P_X^{-1}(\psi_{I,0} - \widehat{\Pi})\right),\\ &= \psi_{I,0} - \left(1 - \frac{\lambda_U}{1-\lambda_I^2}\right)\frac{Z_N}{\alpha_I} - \left(1 - \frac{\lambda_I\lambda_U}{1-\lambda_I^2}\right)(\psi_{I,0} - \widehat{\Pi}),\\ &= \psi_{I,0} - \frac{\lambda_I}{1+\lambda_I}\frac{Z_N}{\alpha_I} - \frac{1}{1+\lambda_I}\left(\psi_{I,0} - \widehat{\Pi}\right).\end{aligned}$$

The result for $\widehat{\psi}_{ns,U,\iota}$ can be deduced from the clearing condition (2), which using the current notation implies that in the limit

$$\widehat{\psi}_{ns,U,\iota} = \frac{1}{1-\lambda_I}\left(\widehat{\Pi} - \lambda_I\frac{Z_N}{\alpha_I} - \lambda_I\widehat{\psi}_{ns,I,\iota}\right).$$

□

Appendix B. Proofs from Sections 3 & 5

Proof of Proposition 3.1. The first statement follows directly from (31) and $\widehat{y} > 0$. For the second, direct calculation shows for P_U from Proposition 2.9 and $P_{U,\iota}$ from (30) that

(51) $$P_U = \frac{1-R}{R}\frac{\beta}{1+\beta}P_X; \qquad P_{U,\iota} = \frac{1-R}{R}\left(\frac{\beta}{(1+\widehat{y})^2+\beta}\right)P_X.$$

Thus, $P_U > P_{U,\iota}$ is equivalent to $1 < (1+\widehat{y})^2$, which holds as $\widehat{y} > 0$. □

Proof of Proposition 3.3. We start with $p(h)$ from (23). From (23), (51) and $\lambda_I + \lambda_U = 1$ we obtain

$$P = \frac{(1-R)(\beta+\lambda_I)}{R(1+\beta)}P_X; \quad P + P_X = \frac{\lambda_U R + \beta + \lambda_I}{R(1+\beta)}P_X; \quad (P_X+P)^{-1}P = \frac{(1-R)(\beta+\lambda_I)}{\lambda_U R + \beta + \lambda_I}1_d,$$

and hence the formula for $p(h)$. We next consider $p_\iota(h)$ from (18) and drop the explicit dependence on $\widehat{y}$ in $M, \mathcal{M}, \mathcal{V}, V$. First, (30) implies $M\mathcal{M} = (1-R)\widehat{y}/(1+2\widehat{y})1_d$ which gives the $h_\iota - p_0$ term. As for the $\psi_{I,0} - \widehat{\Pi}$ term, using (17), (51) we find

(52) $$P_\iota = \frac{(1-R)(\beta + \lambda_I(1+\widehat{y})^2)}{R(\beta + (1+\widehat{y})^2)}P_X.$$

This implies

$$\begin{aligned}P_\iota^{-1} + P_X^{-1} &= \frac{\beta + (\lambda_U R + \lambda_I)(1+\widehat{y})^2}{(1-R)(\beta + \lambda_I(1+\widehat{y})^2)}P_X^{-1},\\ P_X^{-1} - M\mathcal{M}\left(P_\iota^{-1} + P_X^{-1}\right) &= \frac{(1+\widehat{y})\left(\beta + \lambda_I(1+\widehat{y})^2 - \lambda_U R\widehat{y}(1+\widehat{y})\right)}{(1+2\widehat{y})(\beta + \lambda_I(1+\widehat{y})^2)}P_X^{-1},\end{aligned}$$

as well as

(53) $$P_I - P_\iota = \frac{(1-R)\lambda_U(1+\widehat{y})^2}{R(\beta + (1+\widehat{y})^2)}P_X.$$

Therefore,

$$\left(P_X^{-1} - M\mathcal{M}\left(P_\iota^{-1} + P_X^{-1}\right)\right)P_\iota(P_I - P_\iota)^{-1} = \frac{\beta + \lambda_I(1+\widehat{y})^2 - \lambda_U R\widehat{y}(1+\widehat{y})}{\lambda_U(1+2\widehat{y})(1+\widehat{y})}.$$

(49) implies $\lambda_U R\widehat{y}(1+\widehat{y}) = \lambda_I(1+\widehat{y}) + \lambda_U\beta + \lambda_I\beta/(1+\widehat{y})$ and hence

$$\beta + \lambda_I(1+\widehat{y})^2 - \lambda_U R\widehat{y}(1+\widehat{y}) = \frac{\lambda_I\widehat{y}(\beta+(1+\widehat{y})^2)}{1+\widehat{y}},$$

which gives the result.

□

Proof of Proposition 3.4. We retain the notation in (48) and suppress $\widehat{y}$ from the $M, \mathcal{M}, \mathcal{V}, V$ functions. First, direct calculation using (33) shows

$$\frac{2}{1-R}(m_g - m_{g,\iota}) = \frac{1}{1+2\widehat{y}} + \frac{\beta+\lambda_I-\lambda_U R}{\beta+\lambda_I+\lambda_U R}.$$

This gives the result when $\beta+\lambda_I > \lambda_U R$. When $\lambda_U R > \beta+\lambda_I$ define $\widetilde{y}$ through

$$\frac{1}{1+2\widetilde{y}} = \frac{\lambda_U R - (\beta+\lambda_I)}{\lambda_U R+\beta+\lambda_I}.$$

Thus, $m_g > m_{g,\iota}$ if and only if $\widehat{y} < \widetilde{y}$. In the proof of Proposition 2.7 we showed if we define g by the right side of (49), then g is strictly decreasing at $\widehat{y}$. Thus, if $g(\widetilde{y}) < 0$ it must be that $\widehat{y} < \widetilde{y}$. Indeed, $\widehat{y} = \widetilde{y}$ is not possible, and if $\widehat{y} > \widetilde{y}$ then there must be some $\check{y} > 0$ with $g(\check{y}) = 0$, but by the uniqueness statement in Proposition 2.7 we know this is not possible as well.

It therefore suffices to show that $g(\widetilde{y}) < 0$. To this end, write $p := \lambda_U R$ and $q := \beta+\lambda_I$ so that by assumption $p > q$ and $\widetilde{y} = q/(p-q)$ and $\widetilde{y}+1 = p/(p-q)$. In (49) we obtain

$$g(\widetilde{y}) = \frac{p^2}{(p-q)^2}\left(1 - \frac{pq}{\lambda_I(p-q)}\right) + \frac{\lambda_U p\beta}{\lambda_I(p-q)} + \beta.$$

As the common denominator $\lambda_I(p-q)^3$ is positive, we need only show the numerator is negative. The numerator is

$$\lambda_I p^2(p-q) - p^3 q + \lambda_U p\beta(p-q)^2 + \beta\lambda_I(p-q)^3.$$

If we group terms by powers of p the cubic terms vanish, leaving

$$-\left(2\beta+\lambda_I(1+\beta)\right)qp^2 + (\lambda_U+3\lambda_I)\beta q^2 p - \beta\lambda_I q^3. \tag{54}$$

Since $p > q$, the derivative of the above expression is bounded above by $-q^2(3\beta+2\lambda_I) < 0$. Thus, (54) is decreasing in p when $p > q$ and hence bounded above by $-q^3(\beta+\lambda_I) < 0$. The numerator is negative, finishing the result. □

Proof of Proposition 3.5. We again retain the notation in (48) and suppress $\widehat{y}$ from the $M, \mathcal{M}, \mathcal{V}, V$ functions. Using (23), (30) and (51) calculation shows

$$m_{\widehat{\chi}} = \frac{\beta+\lambda_I}{\lambda_U}P_X^{-1}; \qquad m_{\widehat{\chi},\iota} = R\widehat{y}P_X^{-1}.$$

Therefore, $m_{\widehat{\chi}} > m_{\widehat{\chi},\iota}$ if and only if $\widehat{y} < \widetilde{y} := (\beta+\lambda_I)/(\lambda_U R)$. We will show $g(\widetilde{y}) < 0$ for g defined by the right side of (49), and this will give the result, as the proof of Proposition 2.7 showed for $y > 0$ that $g(y) < 0$ if and only if $y > \widehat{y}$. To this end, from (49)

$$\begin{aligned} g(\widetilde{y}) &= \frac{(\lambda_U R+\beta+\lambda_I)^2}{\lambda_U^2 R^2}\left(1 - \frac{\beta+\lambda_I}{\lambda_I}\right) + \frac{\beta(\lambda_U R+\beta+\lambda_I)}{R\lambda_I} + \beta, \\ &= \frac{-\beta}{\lambda_I\lambda_U^2 R^2}\left((\lambda_U R+\beta+\lambda_I)^2 - \lambda_U^2 R(\lambda_U R+\beta+\lambda_I) - \lambda_I\lambda_U^2 R^2\right), \\ &= \frac{-\beta}{\lambda_I\lambda_U^2 R^2}\left(\lambda_U R(\beta+\lambda_I)(1+\lambda_I) + (\beta+\lambda_I)^2\right), \end{aligned}$$

where we have used that $\lambda_I + \lambda_U = 1$. The result follows as $\beta, \lambda_I, \lambda_U, R > 0$. □

Proof of Proposition 5.1. The expected value in (43) is negative if and only if

$$\widehat{y} > \overline{y} := \frac{(1+p_I)(\lambda+\kappa p_I)}{2(1-\lambda)}. \tag{55}$$

Next, as shown in the proof of Proposition 3.4, if we define the function g by the right side of (29) then $\widehat{y} > \overline{y}$ if and only if $g(\overline{y}) > 0$ (since function g is decreasing). Easy calculations then show

$$g(\overline{y}) = \frac{(\lambda-\kappa p_I)\left((\lambda+\kappa p_I)(1+p_I)+2(1-\lambda)\right)^2}{8(1-\lambda)^2\lambda} + \frac{\kappa p_I((1+p_I)(\lambda+\kappa p_I)+2)}{2\lambda}. \tag{56}$$

The sign of the right hand side above coincides with that of

$$\begin{aligned}
&(\lambda-\kappa p_I)\left((\lambda+\kappa p_I)(1+p_I)+2(1-\lambda)\right)^2 + 4(1-\lambda)^2\kappa p_I((1+p_I)(\lambda+\kappa p_I)+2)\\
&\quad = (\lambda+\kappa p_I)\left(((1+p_I)\lambda+2(1-\lambda))^2 - 4\lambda(1-\lambda)\kappa p_I(1+p_I) - \kappa^2 p_I^2(1+p_I)^2\right).
\end{aligned}$$

Therefore $\widehat{y} > \overline{y}$ if and only if

$$\kappa < \overline{\kappa} := \frac{1}{p_I(1+p_I)}\left(\sqrt{(2\lambda(1-\lambda))^2 + (\lambda(1+p_I)+2(1-\lambda))^2} - 2\lambda(1-\lambda)\right).$$

Using (28) we see that as the insider becomes very risk averse ($\alpha_I \to 0$) we have $\lambda \to 0$ and hence $\overline{\kappa} \to 2/(p_I(1+p_I))$ while $\kappa \to 0$. Thus $\widehat{y} > \overline{y}$. On the other hand, as the insider becomes sufficiently risk tolerant ($\alpha_I \to \infty$), $\overline{\kappa} \to 1/p_I$ but $\kappa \to \infty$. Thus, in this instance $\widehat{y} < \overline{y}$. □

Appendix C. Welfare formulas used in Section 4

We use the notation (48) throughout. We begin by identifying interim certainty equivalents in the price taking case. From (20), the insider has certainty equivalent function (recall (21) implies $\sigma(G,H) = \sigma(G,Z_N)$)

$$\begin{aligned}
\frac{1}{\alpha_I}\mathrm{CE}_0^I(g,z) &= -\log\left(\mathbb{E}\left[e^{-\psi_{I,0}' p(h) - \widehat{\psi}_I(G,Z_N)'(X-p(H))}\big|\sigma(G,Z_N)\right]\right)\big|_{(G,Z_N)=(g,z)},\\
&= \psi_{I,0}' p(h) - \frac{1}{2}\widehat{\psi}_I(g,z)'(P_I+P_X)^{-1}\widehat{\psi}_I(g,z)\\
&\qquad + \widehat{\psi}_I(g,z)'(P_I+P_X)^{-1}\left(P_X\mu_X + P_I g - (P_I+P_X)p(h)\right),
\end{aligned}$$

where we used (45) and $X \sim N(\mu_X, P_X^{-1})$. Therefore, (24) implies

$$\frac{1}{\alpha_I}\mathrm{CE}_0^I(g,z) = \psi_{I,0}' p(h) + \frac{1}{2}\left\|(P_I+P_X)^{-1/2}\widehat{\psi}_I(g,z)\right\|^2. \tag{57}$$

The uninformed has the same signal type as the insider, giving

$$\frac{1}{\alpha_U}\mathrm{CE}_0^U(h) = \psi_{U,0}' p(h) + \frac{1}{2}\left\|(P_U+P_X)^{-1/2}\widehat{\psi}_U(h)\right\|^2, \tag{58}$$

where $\widehat{\psi}_U$ is from (24). In the price impact equilibrium (recall (10) implies $\sigma(G,H_\iota) = \sigma(G,Z_N)$) we first obtain

$$\frac{1}{\alpha_I}\mathrm{CE}_{0,\iota}^I(g,z) = -\log\left(\mathbb{E}\left[e^{-\psi_{I,0}' p_\iota(\widehat{\psi}_I,Z_N) - \widehat{\psi}_{I,\iota}'(X - p_\iota(\widehat{\psi}_I,Z_N))}\big|\sigma(G,Z_N)\right]\right)\big|_{(G,Z_N)=(g,z)}.$$

Using (5) and (45), we see that $-(1/\alpha_I)\mathrm{CE}_{0,\iota}^I(g,z)$ is (46), evaluated at $\widehat{\psi}_{I,\iota}$. Therefore

$$\begin{aligned}
\frac{1}{\alpha_I}\mathrm{CE}_{0,\iota}^I(g,z) &= -\psi_{I,0}' M\left(\psi_{I,0} - \frac{z}{\alpha_I}\right) + \psi_{I,0}' V\\
&\qquad + \frac{1}{2}\left\|\left(M + M' + (P_I+P_X)^{-1}\right)^{1/2}\widehat{\psi}_{I,\iota}(g,z)\right\|^2,
\end{aligned} \tag{59}$$

which below will be further specified plugging in for M from (7) and $\widehat{\psi}_{I,\iota}$ from (47). For the uninformed agent U, as she is a price taker, the formulas are the same as in the non-impact case except with $H_\iota, P_{U,\iota}, p_\iota$ replacing H, P_U, p. As such

$$\frac{1}{\alpha_U}\mathrm{CE}^U_{0,\iota}(h_\iota) = \psi'_{U,0}p_\iota(h_\iota) + \frac{1}{2}\left\|(P_{U,\iota}+P_X)^{-1/2}\,\widehat{\psi}_{U,\iota}(h_\iota)\right\|^2, \tag{60}$$

where $\widehat{\psi}_{U,\iota}$ is from (13). Using the above general formulas, the first Lemma identifies interim welfare in the price taking equilibrium under Assumption 2.6 and using the notation in (48).

Lemma C.1.

$$\frac{1}{\alpha_I}CE^I_0(g,z) = \psi'_{I,0}\mu_X + \frac{(1-R)(\beta+\lambda_I)}{\lambda_U R+\beta+\lambda_I}\psi'_{I,0}P_X^{-1/2}\Bigg(P_X^{1/2}(g-\mu_X) + \frac{R}{1-R}P_X^{-1/2}\frac{z}{\alpha_I} - \frac{R(1+\beta)}{(1-R)(\beta+\lambda_I)}P_X^{-1/2}\widehat{\Pi}\Bigg) + \frac{\lambda_U^2 R(1-R)^2}{2(\lambda_U R+\beta+\lambda_I)^2}\left\|P_X^{1/2}(g-\mu_X) - \frac{\beta+\lambda_I}{\lambda_U(1-R)}P_X^{-1/2}\frac{z}{\alpha_I} + \frac{1+\beta}{\lambda_U(1-R)}P_X^{-1/2}\widehat{\Pi}\right\|^2.$$

And

$$\frac{1}{\alpha_U}CE^U_0(h) = \psi'_{U,0}\mu_X + \frac{(1-R)(\beta+\lambda_I)}{\lambda_U R+\beta+\lambda_I}\psi'_{U,0}P_X^{-1/2}\Bigg(P_X^{1/2}(h-\mu_X) - \frac{R(1+\beta)}{(1-R)(\beta+\lambda_I)}P_X^{-1/2}\widehat{\Pi}\Bigg) + \frac{R(1+\beta)\lambda_I^2(1-R)^2}{2(R+\beta)(\lambda_U R+\beta+\lambda_I)^2}\left\|P_X^{1/2}(h-\mu_X) - \frac{R+\beta}{\lambda_I(1-R)}P_X^{-1/2}\widehat{\Pi}\right\|^2.$$

Proof of Lemma C.1. Proposition 3.3 and (15) imply $p(h) = \mu_X - P_X^{-1}\widehat{\Pi} + m_g\left(h-\mu_X+P_X^{-1}\widehat{\Pi}\right)$, where m_g is from (33). Therefore, from (21) and (24) we find

$$\widehat{\psi}_I(g,h) = P_X\mu_X + P_I g - (P_I+P_X)\left(\mu_X - P_X^{-1}\widehat{\Pi} + m_g\left(g + P_I^{-1}\frac{z}{\alpha_I} - \mu_X + P_X^{-1}\widehat{\Pi}\right)\right).$$

Grouping terms by $g-\mu_X$, z/α_I and $\widehat{\Pi}$ we obtain

$$\begin{aligned} g-\mu_X:&\quad P_I - (P_I+P_X)m_g = \frac{\lambda_U(1-R)}{\lambda_U R+\beta+\lambda_I}P_X,\\ \frac{z}{\alpha_I}:&\quad -(P_I+P_X)m_g P_I^{-1} = -\frac{\beta+\lambda_I}{\lambda_U R+\beta+\lambda_I}1_d,\\ \widehat{\Pi}:&\quad (P_I+P_X)P_X^{-1} - (P_I+P_X)m_g P_X^{-1} = \frac{1+\beta}{\lambda_U R+\beta+\lambda_I}1_d. \end{aligned}$$

Therefore

$$\widehat{\psi}_I(g,z) = \frac{1}{\lambda_U R+\beta+\lambda_I}\left(\lambda_U(1-R)P_X\,(g-\mu_X) - (\beta+\lambda_I)\frac{z}{\alpha_I} + (1+\beta)\widehat{\Pi}\right). \tag{61}$$

Similar calculations show

$$p(h) = \mu_X + \frac{(1-R)(\beta+\lambda_I)}{\lambda_U R+\beta+\lambda_I}\left(g-\mu_X + \frac{R}{1-R}P_X^{-1}\frac{z}{\alpha_I} - \frac{R(1+\beta)}{(1-R)(\beta+\lambda_I)}P_X^{-1}\widehat{\Pi}\right). \tag{62}$$

As $(P_I+P_X)^{-1} = RP_X^{-1}$ we obtain from (57)

$$\frac{1}{\alpha_I}\mathrm{CE}^I_0(g,z) = \psi'_{I,0}\mu_X + \frac{(1-R)(\beta+\lambda_I)}{\lambda_U R+\beta+\lambda_I}\psi'_{I,0}\left(g-\mu_X + \frac{R}{1-R}P_X^{-1}\frac{z}{\alpha_I} - \frac{R(1+\beta)}{(1-R)(\beta+\lambda_I)}P_X^{-1}\widehat{\Pi}\right) + \frac{R\lambda_U^2(1-R)^2}{2(\lambda_U R+\beta+\lambda_I)^2}\left\|P_X^{1/2}(g-\mu_X) - \frac{\beta+\lambda_I}{\lambda_U(1-R)}P_X^{-1/2}\frac{z}{\alpha_I} + \frac{1+\beta}{\lambda_U(1-R)}P_X^{-1/2}\widehat{\Pi}\right\|^2.$$

The result follows factoring out $P_X^{-1/2}$ from the linear expression. The calculations are similar for the uniformed trader. From (21) and (24) we obtain

$$\widehat{\psi}_U(h) = P_X\mu_X + P_U h - (P_U + P_X)\left(\mu_X - P_X^{-1}\widehat{\Pi} + m_g\left(h - \mu_X + P_X^{-1}\widehat{\Pi}\right)\right).$$

From (51) we see $P_U + P_X = (R+\beta)/(R(1+\beta))P_X$, and hence grouping terms by $h - \mu_X$ and $\widehat{\Pi}$ we obtain

$$\begin{aligned} h-\mu_X: &\quad P_U - (P_U+P_X)m_g = -\frac{\lambda_I(1-R)}{\lambda_U R+\beta+\lambda_I}P_X, \\ \widehat{\Pi}: &\quad (1-m_g)\left(P_U+P_X\right)P_X^{-1} = \frac{R+\beta}{\lambda_U R+\beta+\lambda_I}1_d. \end{aligned}$$

Therefore,

$$\widehat{\psi}_U(h) = -\frac{\lambda_I(1-R)}{\lambda_U R+\beta+\lambda_I}P_X(h-\mu_X) + \frac{R+\beta}{\lambda_U R+\beta+\lambda_I}\widehat{\Pi}.$$

As $(P_U+P_X)^{-1} = R(1+\beta)/(R+\beta)P_X^{-1}$, we conclude from (31), (58) and (62)

$$\begin{aligned} \frac{1}{\alpha_U}\mathrm{CE}_0^U(h) &= \psi_{U,0}'\mu_X + \frac{(1-R)(\beta+\lambda_I)}{\lambda_U R+\beta+\lambda_I}\psi_{U,0}'\left(h-\mu_X - \frac{R(1+\beta)}{(1-R)(\beta+\lambda_I)}P_X^{-1}\widehat{\Pi}\right) \\ &\quad + \frac{R(1+\beta)\lambda_I^2(1-R)^2}{2(R+\beta)(\lambda_U R+\beta+\lambda_I)^2}\left\|P_X^{1/2}(h-\mu_X) - \frac{R+\beta}{\lambda_I(1-R)}P_X^{-1/2}\widehat{\Pi}\right\|^2. \end{aligned}$$

The result follows factoring out $P_X^{-1/2}$ from the linear expression.

□

The next Lemma identifies interim welfare in the price impact equilibrium.

Lemma C.2.

$$\begin{aligned} \frac{1}{\alpha_I}CE_{0,\iota}^I(g,z) &= \psi_{I,0}'\mu_X + \psi_{I,0}'P_X^{-1/2}\Bigg(\widehat{y}RP_X^{-1/2}\frac{z}{\alpha_I} - \frac{(\widehat{y}+1)^2+\beta}{\lambda_U(\widehat{y}+1)^2}P_X^{-1/2}\widehat{\Pi} \\ &\qquad - \frac{\beta\widehat{y}}{(\widehat{y}+1)^2}P_X^{-1/2}\psi_{I,0}\Bigg) \\ &\quad + \frac{(1-R)^2}{2R(1+2\widehat{y})}\Bigg\|P_X^{1/2}(g-\mu_X) - \frac{\widehat{y}R}{1-R}P_X^{-1/2}\frac{z}{\alpha_I} + \frac{(\widehat{y}+1)^2+\beta}{\lambda_U(1-R)(\widehat{y}+1)^2}P_X^{-1/2}\widehat{\Pi} \\ &\qquad + \frac{\widehat{y}\left(R(\widehat{y}+1)^2+\beta\right)}{(1-R)(\widehat{y}+1)^2}P_X^{-1/2}\psi_{I,0}\Bigg\|^2. \end{aligned}$$

And

$$\begin{aligned} \frac{1}{\alpha_U}CE_{0,\iota}^U(h_\iota) &= \psi_{U,0}'\mu_X + \psi_{U,0}'P_X^{-1/2}\Bigg(\frac{(1-R)\widehat{y}}{1+2\widehat{y}}P_X^{1/2}(h_\iota-\mu_X) \\ &\quad + \frac{\widehat{y}\lambda_I(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(\widehat{y}+1)^2}P_X^{-1/2}\psi_{I,0} - \frac{\beta+(1+\widehat{y})^2}{\lambda_U(1+2\widehat{y})(1+\widehat{y})}P_X^{-1/2}\widehat{\Pi}\Bigg) \\ &\quad + \frac{1}{2}\frac{\lambda_I^2(1-R)^2(\beta+(1+\widehat{y})^2)}{\lambda_U^2R(1+2\widehat{y})^2(\beta+R(1+\widehat{y})^2)}\Bigg\|P_X^{1/2}(h_\iota-\mu_X) + \frac{(\beta+R(1+\widehat{y})^2)\widehat{y}}{(1+\widehat{y})^2(1-R)}P_X^{-1/2}\psi_{I,0} \\ &\quad - \frac{\beta+R(1+\widehat{y})^2}{(1+\widehat{y})(1-R)\lambda_I}P_X^{-1/2}\widehat{\Pi}\Bigg\|^2. \end{aligned}$$

Proof of Lemma C.2. We start by analyzing $\widehat{\psi}_{I,\iota}$ from (47), using (7) and (30). This gives

$$\widehat{\psi}_{I,\iota}(g,z) = \frac{1}{1+2\widehat{y}}\left(P_I g + P_X\mu_X - (P_I+P_X)V + 2\widehat{y}\left(\psi_{I,0} - \frac{z}{2\alpha_I}\right)\right).$$

Using the notation in (48), plugging in for V from (50), and recalling (15) we obtain

$$\widehat{\psi}_{I,\iota}(g,z) = \frac{1}{1+2\widehat{y}}\Bigg(\frac{1-R}{R}P_X(g-\mu_X) + \frac{1}{R}\widehat{\Pi} - \frac{\beta+\lambda_I(1+\widehat{y})^2}{\lambda_U R(1+\widehat{y})^2}(\psi_{I,0}-\widehat{\Pi}) + 2\widehat{y}\left(\psi_{I,0} - \frac{z}{2\alpha_I}\right)\Bigg).$$

Grouping terms by $P_X(g-\mu_X), \widehat{\Pi}$ and z/α_I we obtain

$$P_X(g-\mu_X) : \frac{1-R}{R(1+2\widehat{y})}, \qquad \widehat{\Pi} : \frac{(1+\widehat{y})^2+\beta}{\lambda_U R(1+\widehat{y})^2(1+2\widehat{y})}, \qquad \frac{z}{\alpha_I} : -\frac{\widehat{y}}{1+2\widehat{y}}.$$

The $\psi_{I,0}$ terms are

$$\frac{1}{R(1+2\widehat{y})}\left(2R\widehat{y} - \frac{\beta+\lambda_I(1+\widehat{y})^2}{\lambda_U(1+\widehat{y})^2}\right).$$

As $\widehat{y}$ solves (49) we know

$$\frac{\beta+\lambda_I(1+\widehat{y})^2}{\lambda_U(1+\widehat{y})^2} = R\widehat{y} - \frac{\beta\widehat{y}}{(1+\widehat{y})^2}. \tag{63}$$

Plugging this in and simplifying gives the $\psi_{I,0}$ terms

$$\frac{\widehat{y}(\beta+R(1+\widehat{y})^2)}{R(1+2\widehat{y})(1+\widehat{y})^2}.$$

Therefore

$$\begin{aligned}\widehat{\psi}_{I,\iota}(g,z) = {} & \frac{1-R}{R(1+2\widehat{y})}P_X(g-\mu_X) + \frac{(1+\widehat{y})^2+\beta}{\lambda_U R(1+\widehat{y})^2(1+2\widehat{y})}\widehat{\Pi} - \frac{\widehat{y}}{1+2\widehat{y}}\frac{z}{\alpha_I} \\ & + \frac{\widehat{y}(\beta+R(1+\widehat{y})^2)}{R(1+2\widehat{y})(1+\widehat{y})^2}\psi_{I,0}.\end{aligned} \tag{64}$$

Next, specifying M, P_I, P_X from (30) and V from (50) in (59) gives

$$\begin{aligned}\frac{1}{\alpha_I}\mathrm{CE}^I_{0,\iota}(g,z) = {} & -R\widehat{y}\psi'_{I,0}P_X^{-1}\left(\psi_{I,0} - \frac{z}{\alpha_I}\right) + \psi'_{I,0}\Bigg(\mu_X - \frac{(\widehat{y}+1)^2+\beta}{\lambda_U(\widehat{y}+1)^2}P_X^{-1}\widehat{\Pi} \\ & + \frac{(\widehat{y}+1)^2+\beta}{(\widehat{y}+1)^2}\left(\frac{\lambda_I}{\lambda_U} + \frac{\beta}{(\widehat{y}+1)^2+\beta}\right)P_X^{-1}\psi_{I,0}\Bigg) \\ & + \frac{1}{2}R(1+2\widehat{y})\Bigg\|P_X^{-1/2}\Bigg(\frac{1-R}{R(1+2\widehat{y})}P_X(g-\mu_X) + \frac{(1+\widehat{y})^2+\beta}{\lambda_U R(1+\widehat{y})^2(1+2\widehat{y})}\widehat{\Pi} \\ & - \frac{\widehat{y}}{1+2\widehat{y}}\frac{z}{\alpha_I} + \frac{\widehat{y}(\beta+R(1+\widehat{y})^2)}{R(1+2\widehat{y})(1+\widehat{y})^2}\psi_{I,0}\Bigg)\Bigg\|^2.\end{aligned}$$

By factoring out $(1-R)/(R(1+2\widehat{y}))$ we obtain the quadratic term in the statement of the Lemma. We also immediately obtain the $\psi'_{I,0}\mu_X$, $\psi'_{I,0}P_X^{-1}z/\alpha_I$ and $\psi'_{I,0}P_X^{-1}\widehat{\Pi}$ terms. As for the $\psi'_{I,0}P_X^{-1}\psi_{I,0}$ terms, we have

$$-R\widehat{y} + \frac{\lambda_I(\beta+(1+\widehat{y})^2}{\lambda_U(1+\widehat{y})^2} + \frac{\beta}{(1+\widehat{y}^2)} = -\frac{\beta\widehat{y}}{(1+\widehat{y})^2},$$

where the equality follows from (63). This gives the expression for $\mathrm{CE}^I_{0,\iota}(g,z)$. As for the uninformed trader, in view of (60), let us calculate $\widehat{\psi}_{U,\iota}$ from (13) using $p_\iota(h_\iota)$ from (32).

$$\begin{aligned}\widehat{\psi}_{U,\iota}(h_\iota) = {} & P_X\mu_X + P_{U,\iota}h_\iota - (P_{U,\iota}+P_X)\Bigg(\mu_X - P_X^{-1}\widehat{\Pi} + \frac{(1-R)\widehat{y}}{1+2\widehat{y}}\left(h_\iota - \mu_X + P_X^{-1}\widehat{\Pi}\right) \\ & + \frac{\lambda_I\widehat{y}(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}P_X^{-1}\left(\psi_{I,0}-\widehat{\Pi}\right)\Bigg).\end{aligned}$$

Using (30) and (28) we find

$$P_{U,\iota} = \frac{(1-R)\beta}{R(\beta+(1+\widehat{y})^2)}P_X, \qquad P_{U,\iota}+P_X = \frac{\beta+R(1+\widehat{y})^2}{R(\beta+(1+\widehat{y})^2)}P_X.$$

Grouping the $P_X(h_\iota-\mu_X)$ and $\psi_{I,0}$ terms

$$P_X(h_\iota-\mu_X): \frac{(1-R)(1+\widehat{y})(\beta-R\widehat{y}(1+\widehat{y}))}{R(1+2\widehat{y})(\beta+(\widehat{y}+1)^2)}, \qquad \psi_{I,0}: -\frac{\lambda_I\widehat{y}(\beta+R(1+\widehat{y})^2)}{\lambda_U R(1+2\widehat{y})(1+\widehat{y})^2}.$$

We may simplify this, as (49) implies

$$\beta - R\widehat{y}(1+\widehat{y}) = -\frac{\lambda_I(\beta+(1+\widehat{y}^2))}{\lambda_U(1+\widehat{y})},$$

so that the terms are

$$P_X(h_\iota-\mu_X): -\frac{(1-R)\lambda_I}{\lambda_U R(1+2\widehat{y})}, \qquad \psi_{I,0}: -\frac{\lambda_I\widehat{y}(\beta+R(1+\widehat{y})^2)}{\lambda_U R(1+2\widehat{y})(1+\widehat{y})^2}.$$

As for $\widehat{\Pi}$ we have

$$\begin{aligned}&\frac{\beta+R(1+\widehat{y})^2}{R(\beta+(1+\widehat{y})^2)}\left(1-\frac{(1-R)\widehat{y}}{1+2\widehat{y}}+\frac{\lambda_I\widehat{y}(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}\right),\\ &= \frac{\beta+R(1+\widehat{y})^2}{R(\beta+(1+\widehat{y})^2)}\left(\frac{1+R\widehat{y}+\widehat{y}}{1+2\widehat{y}}+\frac{\lambda_I\widehat{y}(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}\right)\end{aligned} \tag{65}$$

Again, using (49)

$$\lambda_U(1+\widehat{y})^2(1+R\widehat{y}) = (1+\widehat{y})^2+\beta(1+\lambda_U\widehat{y}).$$

The $\widehat{\Pi}$ terms are thus

$$\frac{\beta+R(1+\widehat{y})^2}{\lambda_U R(1+\widehat{y})(1+2\widehat{y})}.$$

Therefore,

$$\begin{aligned}\widehat{\psi}_{U,\iota}(h_\iota) &= -\frac{(1-R)\lambda_I}{\lambda_U R(1+2\widehat{y})}P_X(h_\iota-\mu_X)+\frac{\beta+R(1+\widehat{y})^2}{\lambda_U R(1+2\widehat{y})(1+\widehat{y})}\widehat{\Pi}\\ &\quad -\frac{\lambda_I\widehat{y}\left(\beta+R(1+\widehat{y})^2\right)}{\lambda_U R(1+2\widehat{y})(1+\widehat{y})^2}\psi_{I,0}.\end{aligned} \tag{66}$$

From (60) we obtain

$$\begin{aligned}\frac{1}{\alpha_U}\mathrm{CE}^U_{0,\iota}(h_\iota) &= \psi'_{U,0}\Bigg(\mu_X - P_X^{-1}\widehat{\Pi}+\frac{(1-R)\widehat{y}}{1+2\widehat{y}}\left(h-\mu_X+P_X^{-1}\widehat{\Pi}\right)\\ &\qquad +\frac{\lambda_I\widehat{y}(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}P_X^{-1}\left(\psi_{I,0}-\widehat{\Pi}\right)\Bigg)\\ &\quad +\frac{1}{2}\frac{R(\beta+(1+\widehat{y})^2)}{\beta+R(1+\widehat{y})^2}\Bigg\|\frac{(1-R)\lambda_I}{\lambda_U R(1+2\widehat{y})}P_X^{1/2}(h_\iota-\mu_X)-\frac{\beta+R(1+\widehat{y})^2}{\lambda_U R(1+2\widehat{y})(1+\widehat{y})}P_X^{-1/2}\widehat{\Pi}\\ &\quad +\frac{\lambda_I\widehat{y}\left(\beta+R(1+\widehat{y})^2\right)}{\lambda_U R(1+2\widehat{y})(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0}\Bigg\|^2.\end{aligned}$$

The quadratic term in the statement of the lemma now follows. We also immediately obtain the $\psi'_{U,0}\mu_X$, $\psi'_{U,0}P_X^{-1}(h_\iota-\mu_x)$ and $\psi'_{U,0}P_X^{-1}\psi_{I,0}$ terms. As for the $\psi'_{U,0}P_X^{-1}\widehat{\Pi}$ terms, we have

$$-\left(1-\frac{(1-R)\widehat{y}}{1+2\widehat{y}}+\frac{\lambda_I\widehat{y}(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}\right)$$

The term within the parentheses is the same as in (65). This gives $\psi'_{I,0}P_X^{-1}\widehat{\Pi}$ terms

$$-\frac{\beta+(1+\widehat{y})^2}{\lambda_U(1+2\widehat{y})(1+\widehat{y})},$$

finishing the result.

□

We now compute ex-ante welfare for the insider in the two equilibria, using Lemmas C.1 and C.2. Throughout we recall from Section 1, and using (30), (48), that

(67)

$$\begin{aligned}
P_X^{1/2}(G-\mu_X) &= \mathcal{E}_X + \sqrt{\frac{R}{1-R}}\mathcal{E}_I =: \sqrt{\frac{1}{1-R}}\mathcal{E}_G; \qquad \mathcal{E}_G \sim N(0,1_d),\\
P_X^{-1/2}\frac{Z_N}{\alpha_I} &= \sqrt{\frac{1-R}{\beta R}}\mathcal{E}_N,\\
P_X^{1/2}(H-\mu_X) &= \mathcal{E}_X + \sqrt{\frac{R}{1-R}}\left(\mathcal{E}_I + \frac{1}{\sqrt{\beta}}\mathcal{E}_N\right) =: \sqrt{\frac{\beta+R}{\beta(1-R)}}\mathcal{E}_H; \qquad \mathcal{E}_H \sim N(0,1_d),\\
P_X^{1/2}(H_\iota-\mu_X) &= \mathcal{E}_X + \sqrt{\frac{R}{1-R}}\left(\mathcal{E}_I + \frac{\widehat{y}+1}{\sqrt{\beta}}\mathcal{E}_N\right) =: \sqrt{\frac{\beta+(1+\widehat{y})^2R}{\beta(1-R)}}\mathcal{E}_{H_\iota}; \qquad \mathcal{E}_{H_\iota} \sim N(0,1_d),
\end{aligned}$$

where $\mathcal{E}_G$ and $\mathcal{E}_N$ are independent. To compute the ex-ante welfare, we also need the following lemma.

Lemma C.3. *Let $\mathcal{E}_A, \mathcal{E}_B$ be independent $N(0,1_d)$ random variables. For constants C_1, C_2, C_3, C_4 with $C_3>0$, and vectors V_1, V_2, V_3 in $\mathbb{R}^d$*

$$\begin{aligned}
&-\log\left(\mathbb{E}\left[e^{-C_1-V_1'(\mathcal{E}_A+C_2\mathcal{E}_B-V_2)-\frac{1}{2}C_3\|\mathcal{E}_A-C_4\mathcal{E}_B+V_3\|^2}\right]\right)\\
&\quad = C_1 - V_1'V_2 + \frac{d}{2}\log\left(1+C_3+C_3C_4^2\right) + \frac{1}{2}\frac{C_3}{1+C_3+C_3C_4^2}V_3'V_3\\
&\qquad + \frac{C_3(C_2C_4-1)}{1+C_3+C_3C_4^2}V_1'V_3 - \frac{1+C_3C_4^2+C_2^2(1+C_3)+2C_2C_3C_4}{2(1+C_3+C_3C_4^2)}V_1'V_1.
\end{aligned}$$

Proof of Lemma C.3. If $\mathcal{E}\sim N(0,1_k)$, $M\in\mathbb{S}^k$ is such that $1_k+M\in\mathbb{S}^k_{++}$, and $W\in\mathbb{R}^k$ then

$$\log\left(\mathbb{E}\left[e^{-\frac{1}{2}\mathcal{E}'M\mathcal{E}+W'\mathcal{E}}\right]\right) = -\frac{1}{2}\log\left(|1_k+M|\right) + \frac{1}{2}W'(1_k+M)^{-1}W. \tag{68}$$

Next, note that

$$-C_1 - V_1'\left(\mathcal{E}_A+C_2\mathcal{E}_B-V_2\right) - \frac{1}{2}C_3\left\|\mathcal{E}_A-C_4\mathcal{E}_B+V_3\right\|^2 = -K - \frac{1}{2}\mathcal{E}'M\mathcal{E} + W'\mathcal{E},$$

for $K = C_1 - V_1'V_2 + (1/2)C_3V_3'V_3$ and

$$\mathcal{E} = \begin{pmatrix}\mathcal{E}_A\\ \mathcal{E}_B\end{pmatrix}; \quad M = \begin{pmatrix}C_31_d & -C_3C_41_d\\ -C_3C_41_d & C_3C_4^21_d\end{pmatrix}; \quad W = \begin{pmatrix}-V_1-C_3V_3\\ -C_2V_1+C_3C_4V_3\end{pmatrix}.$$

It can easily be checked that $1_{2d}+M\in\mathbb{S}^{2d}_{++}$ with

$$|1_{2d}+M| = (1+C_3+C_3C_4^2)^d; \qquad (1_{2d}+M)^{-1} = \frac{1}{1+C_3+C_3C_4^2}\begin{pmatrix}C_3C_4^21_d & C_3C_41_d\\ C_3C_41_d & C_31_d\end{pmatrix}.$$

Therefore, using (68) we obtain

$$-\log\left(\mathbb{E}\left[e^{-C_1-V_1'(\mathcal{E}_A+C_2\mathcal{E}_B-V_2)-\frac{1}{2}C_3\|\mathcal{E}_A-C_4\mathcal{E}_B+V_3\|^2}\right]\right)$$
$$=K+\frac{1}{2}\log\left(|1_{2d}+M|\right)-\frac{1}{2}W'(1_{2d}+M)^{-1}W.$$

The result follows by plugging in for $K, W, 1_{2d}+M$ and simplifying. □

Using Lemma C.3 we now compute ex-ante welfare. We start in the price taking case.

Proposition C.4.

$$\frac{1}{\alpha_I}CE^I_{0-}=\frac{d}{2}\log\left(1+\frac{(1-R)(\lambda_U^2\beta R+(\beta+\lambda_I)^2)}{\beta(\lambda_U R+\beta+\lambda_I)^2}\right)+\psi_{I,0}'\mu_X-\frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0}$$
$$+\frac{1}{2}\frac{\lambda_U^2\beta R(1+\beta)^2}{\beta(\lambda_U R+\beta+\lambda_I)^2+(1-R)(\lambda_U^2\beta R+(\beta+\lambda_I)^2)}\left\|P_X^{-1/2}\left(\psi_{I,0}-\psi_{U,0}\right)\right\|^2$$

and

$$\frac{1}{\alpha_U}CE^U_{0-}=\frac{d}{2}\log\left(1+\frac{R(1+\beta)\lambda_I^2(1-R)}{\beta(\lambda_U R\beta+\lambda_I)^2}\right)+\psi_{U,0}'\mu_X-\frac{1}{2}\psi_{U,0}'P_X^{-1}\psi_{U,0}$$
$$+\frac{1}{2}\frac{\lambda_I^2 R\beta(1+\beta)(R+\beta)}{\beta(\lambda_U R+\beta+\lambda_I)^2+R(1+\beta)\lambda_I^2(1-R)}\left\|P_X^{-1/2}\left(\psi_{U,0}-\psi_{I,0}\right)\right\|^2.$$

Proof of Proposition C.4. We start with the insider. From Lemma C.1 and using (67), the interim certainty equivalent becomes

$$\frac{1}{\alpha_I}\mathrm{CE}^I_0(G,Z_N)=\psi_{I,0}'\mu_X$$
$$+\frac{\sqrt{1-R}(1+\beta-\lambda_U)}{\lambda_U R+\beta+\lambda_I}\psi_{I,0}'P_X^{-1/2}\left(\mathcal{E}_G+\sqrt{\frac{R}{\beta}}\mathcal{E}_N-\frac{R(1+\beta)}{\sqrt{1-R}(\beta+\lambda_I)}P_X^{-1/2}\widehat{\Pi}\right)$$
$$+\frac{\lambda_U^2 R(1-R)}{2(\lambda_U R+\beta+\lambda_I)^2}\left\|\mathcal{E}_G-\frac{\beta+\lambda_I}{\lambda_U\sqrt{\beta R}}\mathcal{E}_N+\frac{1+\beta}{\lambda_U\sqrt{1-R}}P_X^{-1/2}\widehat{\Pi}\right\|^2.$$

We obtain CE^I_{0-} by applying Lemma C.3, using $\mathcal{E}_A=\mathcal{E}_G$, $\mathcal{E}_B=\mathcal{E}_N$ and

$$C_1=\psi_{I,0}'\mu_X;\quad C_2=\sqrt{\frac{R}{\beta}};\quad C_3=\frac{\lambda_U^2R(1-R)}{(\lambda_U R+\beta+\lambda_I)^2};\quad C_4=\frac{\beta+\lambda_I}{\lambda_U\sqrt{\beta R}},$$
$$V_1=\frac{\sqrt{1-R}(\beta+\lambda_I)}{\lambda_U R+\beta+\lambda_I}P_X^{-1/2}\psi_{I,0},;\quad V_2=\frac{R(1+\beta)}{\sqrt{1-R}(\beta+\lambda_I)}P_X^{-1/2}\widehat{\Pi};\quad V_3=\frac{1+\beta}{\lambda_U\sqrt{1-R}}P_X^{-1/2}\widehat{\Pi}.$$

The remainder of the proof simplifies the expression for CE^I_{0-} from Lemma C.3. We know C_1. Next

$$V_1'V_2=\frac{R(1+\beta)}{\lambda_U R+\beta+\lambda_I}\psi_{I,0}'P_X^{-1}\widehat{\Pi}.$$

Continuing

$$1+C_3+C_3C_4^2=\frac{\beta(\lambda_U R+\beta+\lambda_I)^2+(1-R)(\lambda_U^2\beta R+(\beta+\lambda_I)^2)}{\beta(\lambda_U R+\beta+\lambda_I)^2}.$$

Let us define the numerator above as

$$\mathbf{V}:=\beta(\lambda_U R+\beta+\lambda_I)^2+(1-R)(\lambda_U^2\beta R+(\beta+\lambda_I)^2).$$

With this notation,

$$\begin{aligned}
\frac{C_3}{1+C_3+C_3C_4^2}V_3'V_3 &= \frac{\beta R(1+\beta)^2}{\mathbf{V}}\widehat{\Pi}'P_X^{-1}\widehat{\Pi},\\
\frac{C_3(C_2C_4-1)}{1+C_3+C_3C_4^2}V_1'V_3 &= \frac{R(1-R)\lambda_I(1+\beta)^2(\beta+\lambda_I)}{(\lambda_U R+\beta+\lambda_I)\mathbf{V}}\psi_{I,0}'P_X^{-1}\widehat{\Pi},\\
\frac{1+C_3C_4^2+C_2^2(1+C_3)+2C_2C_3C_4}{1+C_3+C_3C_4^2}V_1'V_1 &= \frac{(1-R)(1+\beta)(\beta+\lambda_I)^2}{\mathbf{V}}\psi_{I,0}'P_X^{-1}\psi_{I,0},
\end{aligned}$$

where to obtain the last equality we used $1+C_3C_4^2+C_2^2(1+C_3)+2C_2C_3C_4=(1+\beta)/\beta$. At this point, we may conclude

$$\begin{aligned}
\frac{1}{\alpha_I}\mathrm{CE}_{0-}^I &= \psi_{I,0}'\mu_X - \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0} + \frac{d}{2}\log\left(1+\frac{(1-R)(\lambda_U^2\beta R+(\beta+\lambda_I)^2)}{\beta(\lambda_U R+\beta\lambda_I)^2}\right)\\
&+\left(\frac{R(1-R)\lambda_I(1+\beta)^2(\beta+\lambda_I)}{(\lambda_U R+\beta+\lambda_I)\mathbf{V}} - \frac{R(1+\beta)}{\lambda_U R+\beta+\lambda_I}\right)\psi_{I,0}'P_X^{-1}\widehat{\Pi}\\
&+\frac{1}{2}\frac{\beta R(1+\beta)^2}{\mathbf{V}}\widehat{\Pi}'P_X^{-1}\widehat{\Pi}\\
&+\frac{1}{2}\left(1-\frac{(1-R)(1+\beta)(\beta+\lambda_I)^2}{\mathbf{V}}\right)\psi_{I,0}'P_X^{-1}\psi_{I,0}.
\end{aligned}$$

The $\psi_{I,0}'P_X^{-1}\widehat{\Pi}$ terms are

$$\frac{R(1+\beta)}{\mathbf{V}(\lambda_U R+\beta+\lambda_I)}\left(\lambda_I(1-R)(1+\beta)(\beta+\lambda_I)-\mathbf{V}\right) = -\frac{\beta R(1+\beta)^2}{\mathbf{V}}.$$

and the $\psi_{I,0}'P_X^{-1}\psi_{I,0}$ terms on the last line are

$$\frac{1}{2\mathbf{V}}\left(\mathbf{V}-(1-R)(1+\beta)(\beta+\lambda_I)^2\right) = \frac{\beta R(1+\beta)^2}{2\mathbf{V}}.$$

Therefore,

$$\begin{aligned}
\frac{1}{\alpha_I}\mathrm{CE}_{0-}^I &= \psi_{I,0}'\mu_X - \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0} + \frac{d}{2}\log\left(1+\frac{(1-R)(\lambda_U^2\beta R+(\beta+\lambda_I)^2)}{\beta(\lambda_U R+\beta+\lambda_I)^2}\right)\\
&+\frac{1}{2}\frac{\beta R(1+\beta)^2}{\mathbf{V}}\left\|P_X^{-1/2}\left(\psi_{I,0}-\widehat{\Pi}\right)\right\|^2.
\end{aligned}$$

The result for I follows because from (2), (4) and (28)

$$\widehat{\Pi} = \lambda_U\psi_{U,0}+\lambda_I\psi_{I,0}. \tag{69}$$

We work similarly for the uniformed trader's case. Again by Lemma C.1 and (67) we have

$$\begin{aligned}
\frac{1}{\alpha_U}\mathrm{CE}_0^U(H) &= \psi_{U,0}'\mu_X + \sqrt{\frac{(1-R)(R+\beta)}{\beta}}\frac{\beta+\lambda_I}{\lambda_U R+\beta+\lambda_I}\psi_{U,0}'P_X^{-1/2}\Bigg(\mathcal{E}_H\\
&-\sqrt{\frac{\beta}{(1-R)(\beta+R)}}\frac{R(1+\beta)}{\beta+\lambda_I}P_X^{-1/2}\widehat{\Pi}\Bigg)\\
&+\frac{1}{2}\frac{R(1+\beta)(\lambda_I)^2(1-R)}{\beta(\lambda_U R+\beta+\lambda_I)^2}\left\|\mathcal{E}_H-\sqrt{\frac{\beta(\beta+R)}{1-R}}\frac{1}{\lambda_I}P_X^{-1/2}\widehat{\Pi}\right\|^2.
\end{aligned}$$

We apply Lemma C.3 again using $\mathcal{E}_A = \mathcal{E}_H$ and the constants

$$C_1 = \psi_{U,0}'\mu_X; \quad C_2 = 0; \quad C_3 = \frac{R(1+\beta)\lambda_I^2(1-R)}{\beta(\lambda_U R + \beta\lambda_I)^2}; \quad C_4 = 0,$$

$$V_1 = \sqrt{\frac{(\beta+R)(1-R)}{\beta}}\frac{\beta+\lambda_I}{\lambda_U R+\beta+\lambda_I}P_X^{-1/2}\psi_{U,0}; \quad V_2 = \sqrt{\frac{\beta}{(\beta+R)(1-R)}}\frac{R(1+\beta)}{\beta+\lambda_I}P_X^{-1/2}\widehat{\Pi},$$

$$V_3 = -\sqrt{\frac{\beta(\beta+R)}{1-R}}\frac{1}{\lambda_I}P_X^{-1/2}\widehat{\Pi}.$$

As $C_2 = C_4 = 0$ there are a number of cancellations and hence

$$\frac{1}{\alpha_U}\mathrm{CE}_{0-}^U = C_1 - V_1'V_2 + \frac{d}{2}\log\left(1+C_3\right) + \frac{1}{2}\frac{C_3}{1+C_3}V_3'V_3 - \frac{C_3}{1+C_3}V_1'V_3 - \frac{1}{2}\frac{1}{1+C_3}V_1'V_1.$$

As with I, the remainder of the proof simplifies the expression for the certainty equivalent. We already know C_1. Next,

$$V_1'V_2 = \frac{R(1+\beta)}{\lambda_U R+\beta+\lambda_I}\psi_{U,0}'P_X^{-1}\widehat{\Pi}.$$

Continuing

$$1 + C_3 = 1 + \frac{R(1+\beta)\lambda_I^2(1-R)}{\beta(\lambda_U R+\beta+\lambda_I)^2} = \frac{\mathbf{V}}{\beta(\lambda_U R+\beta+\lambda_I)^2},$$

where

$$\mathbf{V} := \beta(\lambda_U R+\beta+\lambda_I)^2 + R(1+\beta)\lambda_I^2(1-R).$$

Next,

$$\begin{aligned}
\frac{C_3}{1+C_3}V_3'V_3 &= \frac{R\beta(1+\beta)(R+\beta)}{\mathbf{V}}\widehat{\Pi}'P_X^{-1}\widehat{\Pi},\\
-\frac{C_3}{1+C_3}V_1'V_3 &= \frac{R(1+\beta)(1-R)\lambda_I(R+\beta)(\beta+\lambda_I)}{(\lambda_U R+\beta+\lambda_I)\mathbf{V}}\psi_{U,0}'P_X^{-1}\widehat{\Pi},\\
\frac{1}{1+C_3}V_1'V_1 &= \frac{(1-R)(R+\beta)(\beta+\lambda_I)^2}{\mathbf{V}}\psi_{U,0}'P_X^{-1}\psi_{U,0}.
\end{aligned}$$

At this point, we may conclude

$$\begin{aligned}
\frac{1}{\alpha_U}\mathrm{CE}_{0-}^U &= \psi_{U,0}'\mu_X - \frac{1}{2}\psi_{U,0}'P_X^{-1}\psi_{U,0} + \frac{d}{2}\log\left(1 + \frac{R(1+\beta)\lambda_I^2(1-R)}{\beta(\lambda_U R+\beta+\lambda_I)^2}\right)\\
&\quad + \left(\frac{R(1+\beta)(1-R)\lambda_I(R+\beta)(\beta+\lambda_I)}{(\lambda_U R+\beta+\lambda_I)\mathbf{V}} - \frac{R(1+\beta)}{\lambda_U R+\beta+\lambda_I}\right)\psi_{U,0}'P_X^{-1}\widehat{\Pi}\\
&\quad + \frac{1}{2}\frac{R\beta(1+\beta)(R+\beta)}{\mathbf{V}}\widehat{\Pi}'P_X^{-1}\widehat{\Pi}\\
&\quad + \frac{1}{2}\left(1 - \frac{(1-R)(R+\beta)(\beta+\lambda_I)^2}{\mathbf{V}}\right)\psi_{U,0}'P_X^{-1}\psi_{U,0}.
\end{aligned}$$

The $\psi_{U,0}'P_X^{-1}\widehat{\Pi}$ terms are

$$\frac{R(1+\beta)}{(\lambda_U R+\beta+\lambda_I)\mathbf{V}}\left((1-R)\lambda_I(R+\beta)(\beta+\lambda_I) - \mathbf{V}\right) = -\frac{R\beta(1+\beta)\,(R+\beta)}{\mathbf{V}},$$

and the $\psi_{U,0}'P_X^{-1}\psi_{U,0}$ terms on the last line are

$$\frac{1}{2\mathbf{V}}\left(\mathbf{V} - (1-R)(R+\beta)(\beta+\lambda_I)^2\right) = \frac{\beta R(1+\beta)(\beta+R)}{2\mathbf{V}}.$$

Therefore,

$$\frac{1}{\alpha_U}\mathrm{CE}^U_{0-} = \psi'_{U,0}\mu_X - \frac{1}{2}\psi'_{U,0}P_X^{-1}\psi_{U,0} + \frac{d}{2}\log\left(1+\frac{R(1+\beta)\lambda_I^2(1-R)}{\beta(\lambda_U R+\beta+\lambda_I)^2}\right) + \frac{\beta R(1+\beta)(\beta+R)}{2\mathbf{V}}\left\|P_X^{-1/2}(\psi_{U,0}-\widehat{\Pi})\right\|^2,$$

which in view of (69) gives the result for U.

□

Lastly, we consider when insider internalizes price impact.

Proposition C.5.

$$\frac{1}{\alpha_I}CE^I_{0-,\iota} = \psi'_{I,0}\mu_X - \frac{1}{2}\psi'_{I,0}P_X^{-1}\psi_{I,0} + \frac{d}{2}\log\left(1+\frac{(1-R)(\beta+\widehat{y}^2R)}{\beta R(1+2\widehat{y})}\right) + \frac{1}{2}\frac{\beta((\widehat{y}+1)^2+\beta)^2}{(\widehat{y}+1)^4\left(\beta R(1+2\widehat{y})+(1-R)\left(\beta+\widehat{y}^2R\right)\right)}\left\|P_X^{-1/2}\left(\psi_{I,0}-\psi_{U,0}\right)\right\|^2$$

and

$$\frac{1}{\alpha_U}CE^U_{0-,\iota} = \psi'_{U,0}\mu_X - \frac{1}{2}\psi'_{U,0}P_X^{-1}\psi_{U,0} + \log\left(1+\frac{\lambda_I^2(1-R)(\beta+(1+\widehat{y})^2)}{\lambda_U^2\beta R(1+2\widehat{y})^2}\right) + \frac{1}{2}\frac{\beta\lambda_I^2(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)}{(\widehat{y}+1)^4(\lambda_U^2\beta R(1+2\widehat{y})^2+\lambda_I^2(1-R)(\beta+(1+\widehat{y})^2))}\left\|P_X^{-1/2}\left(\psi_{I,0}-\psi_{U,0}\right)\right\|^2.$$

Proof of Proposition C.5. We start with the insider. From Lemma C.2 and (67) the interim certainty equivalent becomes

$$\frac{1}{\alpha_I}\mathrm{CE}^I_{0,\iota}(G,Z_N) = \psi'_{I,0}\mu_X + \widehat{y}\sqrt{\frac{R(1-R)}{\beta}}\psi'_{I,0}P_X^{-1/2}\left(\mathcal{E}_N - \sqrt{\frac{\beta}{R(1-R)}}\frac{\beta+(\widehat{y}+1)^2}{\lambda_U\widehat{y}(\widehat{y}+1)^2}P_X^{-1/2}\widehat{\Pi} - \sqrt{\frac{\beta}{R(1-R)}}\frac{\beta}{(\widehat{y}+1)^2}P_X^{-1/2}\psi_{I,0}\right) + \frac{(1-R)\widehat{y}^2}{\beta(1+\widehat{y})}\left\|\mathcal{E}_N - \sqrt{\frac{\beta}{R}}\frac{1}{\widehat{y}}\mathcal{E}_G - \sqrt{\frac{\beta}{R(1-R)}}\frac{\beta+(\widehat{y}+1)^2}{\lambda_U\widehat{y}(1+\widehat{y})^2}P_X^{-1/2}\widehat{\Pi} - \sqrt{\frac{\beta}{R(1-R)}}\frac{\beta+R(\widehat{y}+1)^2}{(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0}\right\|^2.$$

We now use Lemma C.3, but unlike in Proposition C.4, we set $\mathcal{E}_A=\mathcal{E}_N$ and $\mathcal{E}_B=\mathcal{E}_G$. This gives

$$C_1 = \psi'_{I,0}\mu_X;\quad C_2=0;\quad C_3=\frac{(1-R)\widehat{y}^2}{\beta(1+2\widehat{y})};\quad C_4=\sqrt{\frac{\beta}{R}}\frac{1}{\widehat{y}},$$

$$V_1 = \sqrt{\frac{R(1-R)}{\beta}}\widehat{y}P_X^{-1/2}\psi_{I,0},;\quad V_2=\sqrt{\frac{\beta}{R(1-R)}}\left(\frac{\beta+(\widehat{y}+1)^2}{\lambda_U\widehat{y}(1+\widehat{y})^2}P_X^{-1/2}\widehat{\Pi}+\frac{\beta}{(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0}\right),$$

$$V_3 = -\sqrt{\frac{\beta}{R(1-R)}}\left(\frac{\beta+(\widehat{y}+1)^2}{\lambda_U\widehat{y}(1+\widehat{y})^2}P_X^{-1/2}\widehat{\Pi}+\frac{\beta+R(1+\widehat{y})^2}{(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0}\right).$$

As $C_2=0$ there are a number of cancellations and hence

$$\gamma_U\mathrm{CE}^I_{0-,\iota} = C_1 - V_1'V_2 + \frac{d}{2}\log\left(1+C_3+C_3C_4^2\right) + \frac{1}{2}\frac{C_3}{1+C_3+C_3C_4^2}\|V_1-V_3\|^2 - \frac{1}{2}\|V_1\|^2.$$

As in the price taking case, the remainder of the proof simplifies the expression for the certainty equivalent. We already know C_1. Next

$$V_1'V_2 = \frac{\beta+(1+\widehat{y})^2}{\lambda_U(1+\widehat{y})^2}\psi'_{I,0}P_X^{-1}\widehat{\Pi} + \frac{\beta\widehat{y}}{(1+\widehat{y})^2}\psi'_{I,0}P_X^{-1}\psi_{I,0}.$$

Continuing

$$1 + C_3 + C_3 C_4^2 = 1 + \frac{(1-R)(\beta + R\widehat{y}^2)}{\beta R(1+2\widehat{y})} = \frac{\mathbf{V}}{\beta R(1+2\widehat{y})},$$

where

$$\mathbf{V} := \beta R(1+2\widehat{y}) + (1-R)(\beta + R\widehat{y}^2).$$

Next,

$$\|V_1\|^2 = \frac{R(1-R)\widehat{y}^2}{\beta}\psi_{I,0}' P_X^{-1}\psi_{I,0},$$

$$\begin{aligned}\frac{C_3}{1+C_3+C_3C_4^2}\|V_1 - V_3\|^2 = \frac{\widehat{y}^2}{\beta(1+\widehat{y})^4\mathbf{V}}\Big\| &\left(\beta^2 + R(1+\widehat{y})^2(\beta + (1-R)\widehat{y})\right) P_X^{-1/2}\psi_{I,0} \\ &+ \frac{\beta(\beta+(1+\widehat{y})^2)}{\lambda_U \widehat{y}} P_X^{-1/2}\widehat{\Pi}\Big\|^2.\end{aligned}$$

At this point, we may conclude

$$\begin{aligned}\frac{1}{\alpha_I}\mathrm{CE}^I_{0-,\iota} = \psi_{I,0}'\mu_X - \frac{1}{2}\psi_{I,0}' P_X^{-1}\psi_{I,0} + \log\left(1 + \frac{(1-R)(\beta+R\widehat{y}^2)}{\beta R(1+2\widehat{y})}\right) \\ - \left(\frac{\beta+(1+\widehat{y})^2}{\lambda_U(1+\widehat{y})^2}\psi_{I,0}'P_X^{-1}\widehat{\Pi} + \frac{\beta\widehat{y}}{(1+\widehat{y})^2}\psi_{I,0}'P_X^{-1}\psi_{I,0}\right) - \frac{1}{2}\frac{R(1-R)\widehat{y}^2}{\beta}\psi_{I,0}'P_X^{-1}\psi_{I,0} \\ + \frac{\widehat{y}^2}{2\beta(1+\widehat{y})^4\mathbf{V}}\Big\| \left(\beta^2 + R(1+\widehat{y})^2(\beta+(1-R)\widehat{y})\right) P_X^{-1/2}\psi_{I,0} \\ + \frac{\beta(\beta+(1+\widehat{y})^2)}{\lambda_U\widehat{y}} P_X^{-1/2}\widehat{\Pi}\Big\|^2 + \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0}.\end{aligned}$$

The $\widehat{\Pi}' P_X^{-1}\widehat{\Pi}$ terms are

$$\frac{1}{2}\frac{\beta(\beta+(1+\widehat{y})^2)^2}{\lambda_U^2\mathbf{V}(1+\widehat{y})^4}.$$

The $\psi_{I,0}' P_X^{-1}\widehat{\Pi}$ terms are

$$\begin{aligned}&-\frac{\beta+(1+\widehat{y})^2}{\lambda_U(1+\widehat{y})^2} + \frac{\widehat{y}(\beta+(1+\widehat{y})^2)\left(\beta^2 + R(1+\widehat{y})^2(\beta+\widehat{y}(1-R))\right)}{\lambda_U(1+\widehat{y})^4\mathbf{V}}, \\ &= \frac{\beta+(1+\widehat{y})^2}{\lambda_U(1+\widehat{y})^4\mathbf{V}}\left(-\mathbf{V}(1+\widehat{y})^2 + \beta^2\widehat{y} + R\widehat{y}(1+\widehat{y})^2(\beta+\widehat{y}(1-R))\right).\end{aligned}$$

The terms within the parentheses on the right evaluate to

$$\beta^2\widehat{y} - \beta(1+R\widehat{y})(1+\widehat{y})^2 = -\frac{\beta}{\lambda_U}(\beta + (1+\widehat{y})^2),$$

where the equality follows from (49) and $\lambda_U + \lambda_I = 1$. Therefore, the $\psi_{I,0}' P_X^{-1}\widehat{\Pi}$ terms are

$$-\frac{\beta(\beta+(1+\widehat{y})^2)^2}{\lambda_U^2\mathbf{V}(1+\widehat{y})^4}.$$

Lastly, the $\psi_{I,0}' P_X^{-1}\psi_{I,0}$ terms not on the first line are

$$\begin{aligned}&\frac{1}{2} - \frac{\beta\widehat{y}}{(1+\widehat{y})^2} - \frac{R(1-R)\widehat{y}^2}{2\beta} + \frac{\widehat{y}^2\left(\beta^2 + R(1+\widehat{y})^2(\beta+\widehat{y}(1-R))\right)^2}{2\beta\mathbf{V}(1+\widehat{y})^4}, \\ &= \frac{1}{2} - \frac{\widehat{y}(2\beta^2 + R(1-R)\widehat{y}(1+\widehat{y})^2)}{2\beta(1+\widehat{y})^2} + \frac{\widehat{y}^2\left(\beta^2 + R(1+\widehat{y})^2(\beta+\widehat{y}(1-R))\right)^2}{2\beta\mathbf{V}(1+\widehat{y})^4}.\end{aligned}$$

We create the common denominator $2\beta\mathbf{V}(1+\widehat{y})^4$. The numerator is

$$(1+\widehat{y})^2\mathbf{V}\left(\beta(\widehat{y}+1)^2-\widehat{y}\left(2\beta^2+\widehat{y}R(1-R)(\widehat{y}+1)^2\right)\right) \\ +\widehat{y}^2\left(\beta^2+R(\widehat{y}+1)^2(\beta+\widehat{y}(1-R))\right)^2.$$

Let us group by powers of $(\widehat{y}+1)$. The fourth order terms are

$$\begin{aligned}&\widehat{y}^2R^2(\beta+\widehat{y}(1-R))^2+\mathbf{V}(\beta-\widehat{y}^2R(1-R))\\ &\quad=\widehat{y}^2R^2(\beta+\widehat{y}(1-R))^2+\left(\beta+2\beta R\widehat{y}+(1-R)R\widehat{y}^2\right)(\beta-\widehat{y}^2R(1-R))\\ &\quad=\beta^2\left(1+R\widehat{y}\right)^2.\end{aligned}$$

The second order terms are

$$-2\beta^2\widehat{y}\mathbf{V}+2\beta^2R\widehat{y}^2(\beta+\widehat{y}(1-R))=-2\beta^3\widehat{y}(1+R\widehat{y}).$$

The 0^{th} order term is $\beta^4\widehat{y}^2$. The numerator is thus

$$\beta^2\left((1+\widehat{y})^2(1+R\widehat{y})-\beta\widehat{y}\right)^2=\frac{\beta^2(\beta+(1+\widehat{y})^2)^2}{\lambda_U^2},$$

where the equality follows from (49). Therefore, the $\psi_{I,0}'P_X^{-1}\psi_{I,0}$ terms not on the first line are

$$\frac{\beta(\beta+(1+\widehat{y})^2}{2\lambda_U^2(1+\widehat{y})^4\mathbf{V}}.$$

The result follows for the insider I in view of (69). We next consider U. From Lemma C.2 and (67) the interim certainty equivalent becomes

$$\begin{aligned}\frac{1}{\alpha_U}\mathrm{CE}_{0,\iota}^U&=\psi_{U,0}'\mu_X+\sqrt{\frac{(1-R)(\beta+R(1+\widehat{y})^2)}{\beta}}\frac{\widehat{y}}{1+2\widehat{y}}\psi_{U,0}'P_X^{-1/2}\Bigg(\mathcal{E}_{H_\iota}\\ &\quad+\sqrt{\frac{\beta}{(1-R)(\beta+R(1+\widehat{y})^2)}}\frac{\lambda_I(\beta+(1+\widehat{y})^2)}{\lambda_U(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0}\\ &\quad-\sqrt{\frac{\beta}{(1-R)(\beta+R(1+\widehat{y})^2)}}\frac{\beta+(1+\widehat{y})^2}{\lambda_U\widehat{y}(1+\widehat{y})}P_X^{-1/2}\widehat{\Pi}\Bigg)\\ &\quad+\frac{\lambda_I^2(1-R)(\beta+(1+\widehat{y})^2)}{2\lambda_U^2\beta R(1+2\widehat{y})^2}\Bigg\|\mathcal{E}_{H_\iota}+\sqrt{\frac{\beta(\beta+R(1+\widehat{y})^2))}{1-R}}\frac{\widehat{y}}{(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0}\\ &\quad-\sqrt{\frac{\beta(\beta+R(1+\widehat{y})^2))}{1-R}}\frac{1}{\lambda_I(1+\widehat{y})}P_X^{-1/2}\widehat{\Pi}\Bigg\|^2.\end{aligned}$$

We now use Lemma C.3 with $\mathcal{E}_A=\mathcal{E}_{H_\iota}$ as well as

$$\begin{aligned}&C_1=\psi_{U,0}'\mu_X;\quad C_2=0;\quad C_3=\frac{\lambda_I^2(1-R)(\beta+(1+\widehat{y})^2)}{\lambda_U^2\beta R(1+2\widehat{y})^2};\quad C_4=0,\\ &V_1=\sqrt{\frac{(1-R)(\beta+R(1+\widehat{y})^2)}{\beta}}\frac{\widehat{y}}{1+2\widehat{y}}P_X^{-1/2}\psi_{U,0},\\ &V_2=-\sqrt{\frac{\beta}{(1-R)(\beta+R(1+\widehat{y})^2)}}\left(\frac{\lambda_I(\beta+(1+\widehat{y})^2)}{\lambda_U(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0}-\frac{\beta+(1+\widehat{y})^2}{\lambda_U\widehat{y}(1+\widehat{y})}P_X^{-1/2}\widehat{\Pi}\right),\\ &V_3=\sqrt{\frac{\beta(\beta+R(1+\widehat{y})^2))}{1-R}}\left(\frac{\widehat{y}}{(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0}-\frac{1}{\lambda_I(1+\widehat{y})}P_X^{-1/2}\widehat{\Pi}\right).\end{aligned}$$

As $C_2 = C_4 = 0$ there are a number of cancellations and hence

$$\gamma_U \mathrm{CE}^I_{0-,\iota} = C_1 - V_1'V_2 + \frac{d}{2}\log(1+C_3) + \frac{1}{2}\frac{C_3}{1+C_3}\|V_1 - V_3\|^2 - \frac{1}{2}\|V_1\|^2.$$

As with I, the remainder of the proof simplifies the expression for the certainty equivalent. We already know C_1. Next

$$V_1'V_2 = -\frac{\lambda_I\widehat{y}(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}\psi_{U,0}'P_X^{-1}\psi_{I,0} + \frac{\beta+(1+\widehat{y})^2}{\lambda_U(1+2\widehat{y})(1+\widehat{y})}\psi_{U,0}'P_X^{-1}\widehat{\Pi}.$$

Continuing

$$1 + C_3 = 1 + \frac{\lambda_I^2(1-R)(\beta+(1+\widehat{y})^2)}{\lambda_U^2\beta R(1+2\widehat{y})^2} = \frac{\mathbf{V}}{\lambda_U^2\beta R(1+2\widehat{y})^2},$$

where

$$\mathbf{V} := \lambda_U^2\beta R(1+2\widehat{y})^2 + \lambda_I^2(1-R)(\beta+(1+\widehat{y})^2). \tag{70}$$

Next,

$$\begin{aligned}
\|V_1\|^2 &= \frac{(1-R)\widehat{y}^2(\beta+R(1+\widehat{y})^2)}{\beta(1+2\widehat{y})^2}\psi_{U,0}'P_X^{-1}\psi_{U,0},\\
\frac{C_3}{1+C_3}\|V_1 - V_3\|^2 &= \frac{\beta\lambda_I^2(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)}{\mathbf{V}}\\
&\quad\times\left\| P_X^{-1/2}\left(\frac{(1-R)\widehat{y}}{\beta(1+2\widehat{y})}\psi_{U,0} - \frac{\widehat{y}}{(1+\widehat{y})^2}\psi_{I,0} + \frac{1}{\lambda_I(1+\widehat{y})}\widehat{\Pi}\right)\right\|^2.
\end{aligned}$$

At this point, we may conclude

$$\begin{aligned}
\frac{1}{\alpha_U}\mathrm{CE}^U_{0-,\iota} &= \psi_{U,0}'\mu_X - \frac{1}{2}\psi_{U,0}'P_X^{-1}\psi_{U,0} + \log\left(1 + \frac{\lambda_I^2(1-R)(\beta+(1+\widehat{y})^2)}{\lambda_U^2\beta R(1+2\widehat{y})^2}\right)\\
&\quad + \left(\frac{\lambda_I\widehat{y}(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}\psi_{U,0}'P_X^{-1}\psi_{I,0} - \frac{\beta+(1+\widehat{y})^2}{\lambda_U(1+2\widehat{y})(1+\widehat{y})}\psi_{U,0}'P_X^{-1}\widehat{\Pi}\right)\\
&\quad + \left(\frac{1}{2} - \frac{(1-R)\widehat{y}^2(\beta+R(1+\widehat{y})^2)}{2\beta(1+2\widehat{y})^2}\right)\psi_{U,0}'P_X^{-1}\psi_{U,0}\\
&\quad + \frac{\beta\lambda_I^2(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)}{2\mathbf{V}}\\
&\quad \times\left\| P_X^{-1/2}\left(\frac{(1-R)\widehat{y}}{\beta(1+2\widehat{y})}\psi_{U,0} - \frac{\widehat{y}}{(1+\widehat{y})^2}\psi_{I,0} + \frac{1}{\lambda_I(1+\widehat{y})}\widehat{\Pi}\right)\right\|^2.
\end{aligned}$$

The remainder of the proof shows this value coincides with that in the statement of the Lemma. Here, unlike with I we cannot leave $\widehat{\Pi}$ as is until the very end. Instead we substitute in for $\widehat{\Pi}$ as in

(69). Doing this, and simplifying, we obtain

$$
\begin{aligned}
\frac{1}{\alpha_U}\mathrm{CE}^U_{0-,\iota} = \psi'_{U,0}\mu_X - \frac{1}{2}\psi'_{U,0}P_X^{-1}\psi_{U,0} + \log\left(1 + \frac{\lambda_I^2(1-R)(\beta+(1+\widehat{y})^2)}{\lambda_U^2\beta R(1+2\widehat{y})^2}\right) \\
- \frac{\lambda_I(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2}\psi'_{U,0}P_X^{-1}\psi_{I,0} - \frac{\beta+(1+\widehat{y})^2}{(1+2\widehat{y})(1+\widehat{y})}\psi'_{U,0}P_X^{-1}\psi_{U,0} \\
+ \left(\frac{1}{2} - \frac{(1-R)\widehat{y}^2(\beta+R(1+\widehat{y})^2)}{2\beta(1+2\widehat{y})^2}\right)\psi'_{U,0}P_X^{-1}\psi_{U,0} \\
+ \frac{\beta\lambda_I^2(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)}{2\mathbf{V}} \\
\times \left\| P_X^{-1/2}\left(\left(\frac{(1-R)\widehat{y}}{\beta(1+2\widehat{y})} + \frac{\lambda_U}{\lambda_I)(1+\widehat{y})}\right)\psi_{U,0} + \frac{1}{(1+\widehat{y})^2}\psi_{I,0}\right)\right\|^2,
\end{aligned}
$$

The $\psi'_{I,0}P_X^{-1}\psi_{I,0}$ terms are

$$
\frac{\beta\lambda_I^2(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)}{2\mathbf{V}(1+\widehat{y})^4},
$$

which is consistent with that in the statement of the lemma. As for the $\psi'_{I,0}P_X^{-1}\psi_{U,0}$ terms, we claim

$$
\begin{aligned}
&- \frac{\lambda_I(\beta+(1+\widehat{y})^2)}{\lambda_U(1+2\widehat{y})(1+\widehat{y})^2} + \frac{\beta\lambda_I^2(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)}{\mathbf{V}(1+\widehat{y})^2}\left(\frac{(1-R)\widehat{y}}{\beta(1+2\widehat{y})} + \frac{\lambda_U}{\lambda_I(1+\widehat{y})}\right) \\
&\quad = -\frac{\beta\lambda_I^2(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)}{\mathbf{V}(1+\widehat{y})^4},
\end{aligned}
\tag{71}
$$

which is consistent with the Lemma statement. Indeed, cancelling common terms and re-arranging, (71) is the same as

$$
\frac{1}{\lambda_U(1+2\widehat{y})} = \frac{\beta\lambda_I(\beta+R(1+\widehat{y})^2)}{\mathbf{V}}\left(\frac{(1-R)\widehat{y}}{\beta(1+2\widehat{y})} + \frac{\lambda_U}{\lambda_I(1+\widehat{y})} + \frac{1}{(1+\widehat{y})^2}\right),
$$

or equivalently

$$
\begin{aligned}
&\lambda_U(\beta+R(1+\widehat{y})^2)\Big(\lambda_I(1-R)\widehat{y}(1+\widehat{y})^2 + \lambda_U\beta(1+2\widehat{y})(1+\widehat{y}) + \beta\lambda_I(1+2\widehat{y})\Big) \\
&\quad = (1+\widehat{y})^2\mathbf{V} \\
&\quad = \lambda_U^2\beta R(1+2\widehat{y})^2(1+\widehat{y})^2 + \lambda_I^2(1-R)(\beta+(1+\widehat{y})^2)(1+\widehat{y})^2 \\
&\quad = \lambda_U^2\beta R(1+2\widehat{y})^2(1+\widehat{y})^2 + \lambda_U\lambda_I(1-R)(R(1+\widehat{y})^2\widehat{y} - \beta(1+\widehat{y}))(1+\widehat{y})^2,
\end{aligned}
$$

where the last equality uses (63). There is an $\lambda_U\lambda_I R(1-R)\widehat{y}(1+\widehat{y})^4$ term common to both sides. Eliminating this term, the remaining terms each have a β and λ_U in them, which we can also eliminate. This leaves

$$
\begin{aligned}
&(\beta+R(1+\widehat{y})^2)\Big(\lambda_U(1+2\widehat{y})(1+\widehat{y}) + \lambda_I(1+2\widehat{y})\Big) + \lambda_I(1-R)\widehat{y}(1+\widehat{y})^2 \\
&\quad = \lambda_U R(1+2\widehat{y})^2(1+\widehat{y})^2 - \lambda_I(1-R)(1+\widehat{y})^3.
\end{aligned}
$$

Note that $(1+\widehat{y})^3 + \widehat{y}(1+\widehat{y})^2 = (1+\widehat{y})^2(1+2\widehat{y})$. This allows us to eliminate one copy of $(1+2\widehat{y})$ from both sides, to obtain

$$
\begin{aligned}
&(\beta+R(1+\widehat{y})^2)\Big(\lambda_U(1+\widehat{y}) + \lambda_I\Big) + \lambda_I(1-R)(1+\widehat{y})^2 \\
&\quad = \lambda_U R(1+2\widehat{y})(1+\widehat{y})^2 = \lambda_U R\left((1+\widehat{y})^3 + y(1+\widehat{y})^2\right)
\end{aligned}
$$

There is a common $\lambda_U R(1+\widehat{y})^3$ term. Eliminating this term gives

$$\lambda_I(\beta + R(1+\widehat{y})^2) + \lambda_U\beta(1+\widehat{y}) + \lambda_I(1-R)(1+\widehat{y})^2 = \lambda_U R y(1+\widehat{y})^2.$$

The left side above simplifies to

$$\beta + \lambda_U\beta\widehat{y} + \lambda_I(1+\widehat{y})^2.$$

Using (63) again, the right side above simplifies to

$$\lambda_I((1+\widehat{y})^2 + \beta) + \lambda_U\beta(1+\widehat{y}) = \beta + \lambda_U\beta\widehat{y} + \lambda_I(1+\widehat{y})^2,$$

and hence (71) holds. Lastly, for the $\psi_{U,0}' P_X^{-1}\psi_{U,0}$ terms, we claim

$$\begin{aligned} &\frac{1}{2} - \frac{(1-R)\widehat{y}^2(\beta + R(1+\widehat{y})^2)}{2\beta(1+2\widehat{y})^2} - \frac{\beta + (1+\widehat{y})^2}{(1+2\widehat{y})(1+\widehat{y})} \\ &\qquad + \frac{\beta\lambda_I^2(\beta + (1+\widehat{y})^2)(\beta + R(1+\widehat{y})^2)}{2\mathbf{V}}\left(\frac{(1-R)\widehat{y}}{\beta(1+2\widehat{y})} + \frac{\lambda_U}{\lambda_I(1+\widehat{y})}\right)^2 \\ &\qquad\qquad = \frac{\beta\lambda_I^2(\beta + (1+\widehat{y})^2)(\beta + R(1+\widehat{y})^2)}{2\mathbf{V}(1+\widehat{y})^4}, \end{aligned}$$

which if true, would finish the proof. Let us first focus on the terms with $(1+2\widehat{y})^2$ in the denominator on the left side. Here, using (70) we find

$$\begin{aligned} &-\frac{(1-R)\widehat{y}^2(\beta + R(1+\widehat{y})^2)}{2\beta(1+2\widehat{y})^2} + \frac{\beta\lambda_I^2(\beta + (1+\widehat{y})^2)(\beta + R(1+\widehat{y})^2)(1-R)^2\widehat{y}^2}{2\mathbf{V}\beta^2(1+2\widehat{y})^2} \\ &\qquad = -\frac{\lambda_U^2 R(1-R)\widehat{y}^2(\beta + R(1+\widehat{y})^2)}{2\mathbf{V}}. \end{aligned}$$

Next, let us focus on the terms with $(1+\widehat{y})(1+2\widehat{y})$ in the denominator on the left side.

$$\begin{aligned} &-\frac{\beta + (1+\widehat{y})^2}{(1+2\widehat{y})(1+\widehat{y})} + \frac{\lambda_U(1-R)\lambda_I\widehat{y}(\beta + (1+\widehat{y})^2)(\beta + R(1+\widehat{y})^2)}{\mathbf{V}(1+2\widehat{y})(1+\widehat{y})} \\ &\quad = \frac{\beta + (1+\widehat{y})^2}{\mathbf{V}(1+2\widehat{y})(1+\widehat{y})}\Big(-\lambda_U^2\beta R(1+2\widehat{y})^2 - \lambda_I^2(1-R)(\beta + (1+\widehat{y})^2) \\ &\qquad\quad + \lambda_U\lambda_I(1-R)\widehat{y}(\beta + R(1+\widehat{y})^2)\Big) \end{aligned}$$

Using (63) the terms within the parentheses evaluate to

$$\begin{aligned} &-\lambda_U^2\beta R(1+2\widehat{y})^2 - \lambda_I(1-R)(\lambda_U R\widehat{y}(1+\widehat{y})^2 - \lambda_U\beta(1+\widehat{y})) + \lambda_U\lambda_I(1-R)\widehat{y}(\beta + R(1+\widehat{y})^2) \\ &\quad = \lambda_U\beta(1+2\widehat{y})(\lambda_I(1-R) - \lambda_U R(1+2\widehat{y})), \end{aligned}$$

so that

$$\begin{aligned} &-\frac{\beta + (1+\widehat{y})^2}{(1+2\widehat{y})(1+\widehat{y})} + \frac{\lambda_U(1-R)\lambda_I\widehat{y}(\beta + (1+\widehat{y})^2)(\beta + R(1+\widehat{y})^2)}{\mathbf{V}(1+2\widehat{y})(1+\widehat{y})} \\ &\qquad = \frac{\lambda_U\beta(\beta + (1+\widehat{y})^2)(\lambda_I(1-R) - \lambda_U R(1+2\widehat{y}))}{\mathbf{V}(1+\widehat{y})} \end{aligned}$$

Therefore, we must show

$$\begin{aligned} &\frac{1}{2} - \frac{\lambda_U^2 R(1-R)\widehat{y}^2(\beta + R(1+\widehat{y})^2)}{2\mathbf{V}} + \frac{\lambda_U\beta(\beta + (1+\widehat{y})^2)(\lambda_I(1-R) - \lambda_U R(1+2\widehat{y}))}{\mathbf{V}(1+\widehat{y})} \\ &\qquad + \frac{\beta(\beta + (1+\widehat{y})^2)(\beta + R(1+\widehat{y})^2)}{2\mathbf{V}(1+\widehat{y})^4}\left(\lambda_U^2(1+\widehat{y})^2 - \lambda_I^2\right) = 0. \end{aligned}$$

Let us focus on terms containing $\beta(\beta+(1+\widehat{y})^2)$:

$$\frac{\beta(\beta+(1+\widehat{y})^2)}{2\mathbf{V}(1+\widehat{y})^4}\Big(2\lambda_U(\lambda_I(1-R)-\lambda_U R(1+\widehat{y})-\lambda_U R\widehat{y})(1+\widehat{y})^3 + (\beta+R(1+\widehat{y})^2)(\lambda_U^2(1+\widehat{y})^2-\lambda_I^2)\Big)$$

The terms within the parentheses are

$$\begin{aligned}
&2\lambda_U\lambda_I(1-R)(1+\widehat{y})^3-2\lambda_U^2R(1+\widehat{y})^4-2\lambda_U^2R\widehat{y}(1+\widehat{y})^3+\lambda_U^2\beta(1+\widehat{y})^2-\beta\lambda_I^2\\
&\qquad+\lambda_U^2R(1+\widehat{y})^4-\lambda_I^2R(1+\widehat{y})^2\\
&\quad=-\lambda_U^2R(1+\widehat{y})^4+2\lambda_U\lambda_I(1-R)(1+\widehat{y})^3-2\lambda_U(1+\widehat{y})(\lambda_I(1+\widehat{y})^2+\lambda_I\beta+\lambda_U\beta(1+\widehat{y}))\\
&\qquad+\lambda_U^2\beta(1+\widehat{y})^2-\beta\lambda_I^2+\lambda_U^2R(1+\widehat{y})^4-\lambda_I^2R(1+\widehat{y})^2\\
&\quad=-\lambda_U^2R(1+\widehat{y})^4-2\lambda_U\lambda_IR(1+\widehat{y})^3-(\lambda_U^2\beta+\lambda_I^2R)(1+\widehat{y})^2\\
&\qquad-2\lambda_U\lambda_I\beta(1+\widehat{y})-\beta\lambda_I^2\\
&\quad=-(\beta+R(1+\widehat{y})^2\left(\lambda_U^2(1+\widehat{y})^2+2\lambda_U(1+\widehat{y})\lambda_I+\lambda_I^2\right)\\
&\quad=-(\beta+R(1+\widehat{y})^2)(1+\lambda_U\widehat{y})^2.
\end{aligned}$$

Above, the second equality uses (63). Therefore, the terms containing $\beta(\beta+(1+\widehat{y})^2)$ are

$$-\frac{\beta(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)(1+\lambda_U\widehat{y})^2}{2\mathbf{V}(1+\widehat{y})^4},$$

and we must show

$$0=\frac{1}{2}-\frac{\lambda_U^2R(1-R)\widehat{y}^2(\beta+R(1+\widehat{y})^2)}{2\mathbf{V}}-\frac{\beta(\beta+(1+\widehat{y})^2)(\beta+R(1+\widehat{y})^2)(1+\lambda_U\widehat{y})^2}{2\mathbf{V}(1+\widehat{y})^4},$$

or that

$$\mathbf{V}=(\beta+R(1+\widehat{y})^2)\left(\lambda_U^2R(1-R)\widehat{y}^2+\frac{\beta(\beta+(1+\widehat{y})^2)(1+\lambda_U\widehat{y})^2}{(1+\widehat{y})^4}\right).$$

To show this, we note that (49) implies

$$\beta=\frac{(1+\widehat{y})^2(\lambda_UR\widehat{y}-\lambda_I)}{1+\lambda_U\widehat{y}},$$

and hence from (70) we see

$$\mathbf{V}=(1+\widehat{y})^2\left(\lambda_I^2(1-R)+\frac{(\lambda_UR\widehat{y}-\lambda_I)(\lambda_U^2R(1+2\widehat{y})^2+\lambda_I^2(1-R))}{1+\lambda_Uy}\right),$$

and

$$\begin{aligned}
&(\beta+R(1+\widehat{y})^2)\left(\lambda_U^2R(1-R)\widehat{y}^2+\frac{\beta(\beta+(1+\widehat{y})^2)(1+\lambda_U\widehat{y})^2}{(1+\widehat{y})^4}\right)\\
&\quad=(1+\widehat{y})^2\frac{\lambda_UR\widehat{y}-\lambda_I+R(1+\lambda_U\widehat{y})}{1+\lambda_U\widehat{y}}\Big(\lambda_U^2R(1-R)\widehat{y}^2+(\lambda_UR\widehat{y}-\lambda_I)^2+\\
&\qquad+(\lambda_UR\widehat{y}-\lambda_I)(1+\lambda_U\widehat{y})\Big).
\end{aligned}$$

Thus, we want to show

$$\begin{aligned}\lambda_I^2(1-R)(1+\lambda_U y) &+ (\lambda_U R\widehat{y}-\lambda_I)(\lambda_U^2 R(1+2\widehat{y})^2+\lambda_I^2(1-R))\\ &= (\lambda_U R\widehat{y}-\lambda_I+R(1+\lambda_U\widehat{y}))\Big(\lambda_U^2 R(1-R)\widehat{y}^2+(\lambda_U R\widehat{y}-\lambda_I)^2+\\ &\quad +(\lambda_U R\widehat{y}-\lambda_I)(1+\lambda_U\widehat{y})\Big)\\ &= (2\lambda_U R\widehat{y}+R-\lambda_I)\Big(2\lambda_U^2 R\widehat{y}^2+(-2\lambda_U R\lambda_I+\lambda_U R-\lambda_U\lambda_I)\widehat{y}+\lambda_I^2-\lambda_I\Big)\end{aligned}$$

Here, as we no longer need appeal to (49) or (63) we simply match powers of $\widehat{y}$ on both sides of the equation. The cubic terms are

$$4\lambda_U^3 R^2 \text{ vs. } 4\lambda_U^3 R^2 \qquad \checkmark.$$

The quadratic terms are

$$4\lambda_U^3 R^2-4\lambda_U^2\lambda_I R \text{ vs. } 2\lambda_U^2 R(R-\lambda_I)+2\lambda_U R(-2\lambda_U R\lambda_I)+\lambda_U R-\lambda_U\lambda_I) \qquad \checkmark.$$

The linear terms are

$$\begin{aligned}\lambda_U\lambda_I^2(1-R)+\lambda_U R(\lambda_I^2(1-R)+\lambda_I^2 R)-4\lambda_I\lambda_U^2 R \text{ vs.}\\ 2\lambda_U R(\lambda_I^2-\lambda_I)+(R-\lambda_I)(-2\lambda_U R\lambda_I+\lambda_U R-\lambda_U\lambda_I)\end{aligned}$$

Within this term, the quadratic powers in R are

$$-\lambda_U\lambda_I^2+\lambda_U^3 \text{ vs. } -2\lambda_U\lambda_I+\lambda_U \qquad \checkmark.$$

The linear powers in R are

$$-4\lambda_U^2\lambda_I \text{ vs. } 4\lambda_U\lambda_I^2-4\lambda_U\lambda_I \qquad \checkmark.$$

The constants are

$$\lambda_U\lambda_I^2 \text{ vs. } \lambda_U\lambda_I^2 \qquad \checkmark.$$

Lastly, terms which do not depend on $\widehat{y}$ are

$$\begin{aligned}\lambda_I^2(1-R)-\lambda_I(\lambda_U^2 R+\lambda_I^2(1-R)) = -\lambda_U\lambda_I R+\lambda_U\lambda_I^2 \text{ vs.}\\ R(\lambda_I^2-\lambda_I)-\lambda_I^2(\lambda_I-1) = -\lambda_U\lambda_I R+\lambda_U\lambda_I^2 \qquad \checkmark.\end{aligned}$$

The proof is complete. □

Appendix D. Proofs from Section 4

Proof of Proposition 4.2. Set $c=(1-\lambda)/\lambda$, $p=p_I$ and write $\widehat{y}(p_I)=y(p)$ so that (29) is

$$0=(1+y(p))^2\left(1-\frac{cy(p)}{1+p}\right)+\kappa pc(1+y(p))+\kappa p. \tag{72}$$

Next define the function (c.f. (35))

$$Q(p,y)=\frac{\kappa p(1+p)+y^2}{\kappa(1+p)(1+2y)}$$

so that

$$\partial_p Q(p,y)=\frac{\kappa(1+p)^2-y^2}{\kappa(1+p)^2(1+2y)};\quad \partial_y Q(p,y)=\frac{2(y(1+y)-\kappa p(1+p))}{\kappa(1+p)(1+2y)^2}.$$

Additionally, from (72) we deduce

$$\begin{aligned}0=&\left(\kappa+\kappa c(1+y(p))+\frac{cy(p)(1+y(p))^2}{(1+p)^2}\right)\\ &-\left(\frac{c(1+y(p))^2}{1+p}+2(1+y(p))\left(\frac{cy(p)}{1+p}-1\right)-\kappa cp\right)\partial_p y(p).\end{aligned}$$

Using (72) one can show the quantity in front of $\partial_p y(p)$ is strictly positive so that $y(p)$ is increasing in p and hence

$$\partial_p y(p) = \frac{\kappa + \kappa c(1+y(p)) + \frac{cy(p)(1+y(p))^2}{(1+p)^2}}{\frac{c(1+y(p))^2}{1+p} + 2(1+y(p))\left(\frac{cy(p)}{1+p} - 1\right) - \kappa cp}.$$

By the chain rule

$$\partial_p \phi_\iota(p) = \partial_p Q(p, y(p)) + \partial_y Q(p, y(p)) \partial_p y(p), \tag{73}$$

and we wish to show the right side above is positive for all $p > 0, c > 0, \kappa > 0$. To do this we will change perspective. Namely, from (72) we see that $y(p) \geq (1+p)/c$ and in fact $y(p) = (1+p)/c$ when $\kappa = 0$. Thus, let us substitute $y(p) = (1+p)/c + z(p)$ so that $z(p)$ solves (uniquely over the positive reals) the equation

$$0 = -\frac{cz}{1+p}\left(1 + z + \frac{1+p}{c}\right)^2 + \kappa pc\left(1 + z + \frac{1+p}{c}\right) + \kappa p.$$

Now, fix $p > 0, c > 0$ and think about $z = z(\kappa)$. It is straight-forward to show that z is strictly increasing in κ with extreme values $z(0) = 0$ and $z(\infty) = \infty$. Therefore, there is no loss in generality in fixing $p > 0, c > 0, z > 0$ and setting

$$\kappa = \frac{z(1+p+c+cz)^2}{pc(1+p)(2+p+c+cz)}.$$

Plugging in $y = (1+p)/c + z$ and κ as above we obtain

$$\begin{aligned}
\partial_p Q(p, y(p)) &= \frac{zc(1+p)(1+p+c+cz)^2 - p(1+p+cz)(2+p+c+cz)}{z(1+p)(2+2p+c+2z)(1+p+c+cz)^2},\\
\partial_y Q(p, y(p)) &= \frac{2pc((1+p)(1+p+c+cz) + (1+p+cz))}{z(1+p+c+cz)(2+2p+c+2z)^2},\\
\partial_p y(p) &= \frac{(1+p+c+cz)(2+p+c+cz)(zc(1+p) + p(1+p+cz))}{pc(1+p)(cz + (1+p+c+2cz)(2+p+c+cz))}.
\end{aligned}$$

At this point, if one plugs these values into the right side of (73) and takes a common denominator, the numerator is a sixth order polynomial in z. Furthermore, one can directly verify (e.g. using Mathematica) that each of the coefficients in the polynomial is positive for all $p > 0, c > 0$, giving the result since $z > 0$ as well. $\square$

Proof of Proposition 4.5. From (36) we see that $\mathrm{CE}^U_{0-,\iota} \geq \mathrm{CE}^U_{0-}$ is equivalent to

$$f(y) := \frac{\kappa p_I + (1+\widehat{y})^2}{(1+2\widehat{y})^2} \geq \frac{(1-\lambda)^2(1+\kappa p_I)}{(1+p_I\lambda + \kappa p_I(1+p_I))^2} =: \ell.$$

The map $y \to f(y)$ is strictly decreasing with $f(0) = \kappa p_I + 1$ and $f(\infty) = 1/4$. Furthermore, as ℓ is evidently decreasing in $\lambda \in (0,1)$ we know

$$0 < \ell < \frac{1+\kappa p_I}{(1+\kappa p_I(1+p_I))^2} < f(0),$$

and hence $k \leq 1/4$ implies $\mathrm{CE}^U_{0-,\iota} \geq \mathrm{CE}^U_{0-}$. For $1/4 < k$, the positive root of $\kappa p_I + (1+y)^2 = \ell(1+2y)^2$ is

$$\check{y} = \frac{1 - 2\ell + \sqrt{(4\ell-1)\kappa p_I + \ell}}{4\ell - 1}.$$

As shown in the proof of Proposition 2.7, if we define $g(y)$ by the cubic function in (29), then $g(y) > 0$ for $0 < y < \widehat{y}$ and $g(y) < 0$ for $y > \widehat{y}$. Therefore, if $g(\check{y}) < 0$ then $\check{y} > \widehat{y}$ and

$$\frac{\kappa p_I + (1+\widehat{y})^2}{(1+2\widehat{y})^2} > \frac{\kappa p_I + (1+\check{y})^2}{(1+2\check{y})^2} = \ell,$$

giving the result. It therefore suffices perform the following check

(1) Fix $0 < \lambda < 1$, $\kappa, p > 0$ and let $\ell = \ell(\lambda, \kappa, p_I)$ as above.
(2) If $\ell \leq 1/4$ then $\mathrm{CE}^U_{0-,\iota} \geq \mathrm{CE}^U_{0-}$.
(3) If $\ell > 1/4$ then set $\check{y} = \check{y}(\lambda, \kappa, p_I)$ and $g = g(\check{y})$ as above. If $g(\check{y}) < 0$ then $\mathrm{CE}^U_{0-,\iota} \geq \mathrm{CE}^U_{0-}$.

This check can easily be performed by any software tool and one always obtains that either $\ell \leq 1/4$ or $g(\check{y}) < 0$. □

Proof of Proposition 4.6. First, using (4) and (27) we obtain

$$\begin{aligned} &\lim_{\alpha_I \to 0} \frac{\mathrm{CE}^I_{nsn}}{\alpha_I} = \frac{1}{\alpha_U}\Pi'\mu_X - \frac{1}{\alpha_U^2}\Pi' P_X^{-1}\Pi; \quad \lim_{\alpha_I\to\infty} \mathrm{CE}^I_{nsn} = \Pi'\mu_X, \\ &\lim_{\alpha_U\to 0} \mathrm{CE}^I_{nsn} = \Pi'\mu_X - \frac{1}{2\alpha_I}\Pi' P_X^{-1}\Pi; \quad \lim_{\alpha_U\to\infty} \alpha_U \mathrm{CE}^I_{nsn} = \alpha_I \Pi'\mu_X. \end{aligned} \tag{74}$$

For the first item, we fix α_I and let $\alpha_U \to 0$ which corresponds to $\lambda \to 1$ with κ, p_I fixed. It is easy to see that

$$\frac{(1-\lambda)^2\kappa p_I + (1+p_I)(\lambda+\kappa p_I)^2}{\kappa(1+\lambda p_I + \kappa p_I(1+p_I))^2} \to \frac{1}{\kappa(1+p_I)},$$

which, in view of (74), gives gives that

$$\lim_{\alpha_U\to 0} \mathrm{CE}^I_{0-} = \Pi'\mu_X - \frac{1}{2\alpha_I}\Pi' P_X^{-1}\Pi + \alpha_I \frac{d}{2}\log\left(1 + \frac{1}{\kappa(1+p_I)}\right).$$

In particular, the above means that CE^I_{0-} remains bounded. As for $\mathrm{CE}^I_{0-,\iota}$, (29) implies $(1-\lambda)\widehat{y} \to 1 + p_I$. This gives

$$1 + \frac{\kappa p_I(1+p_I) + \widehat{y}^2}{\kappa(1+p_I)(1+2\widehat{y})} = \frac{1}{1-\lambda}\left(1 - \lambda + \times \frac{(1-\lambda)\kappa p_I(1+p_I)/\widehat{y} + (1-\lambda)\widehat{y}}{\kappa(1+p_I)\,(1/\widehat{y}+2)}\right) \approx \frac{1}{2\kappa(1-\lambda)}.$$

Now since $1 - \lambda = \alpha_U/(\alpha_I + \alpha_U)$, we get that

$$\lim_{\alpha_U\to 0} \frac{\mathrm{CE}^I_{0-,\iota}}{-\log(\alpha_U)} = \alpha_I \frac{d}{2},$$

which in particular implies that $\lim_{\alpha_U\to 0} \mathrm{CE}^I_{0-,\iota} = \infty$. Now keep α_I fixed and let $\alpha_U \to \infty$. This corresponds to $\lambda \to 0$ with κ, p_I fixed. Straight-forward computations show

$$\frac{(1-\lambda)^2\kappa p_I + (1+p_I)(\lambda+\kappa p_I)^2}{\kappa(1+\lambda p_I + \kappa p_I(1+p_I))^2} \to \frac{p_I}{1+\kappa p_I(1+p_I)},$$

which, in view of (74),

$$\lim_{\alpha_U\to\infty} \mathrm{CE}^I_{0-} = \alpha_I \frac{d}{2}\log\left(1 + \frac{p_I}{1+\kappa p_I(1+p_I)}\right).$$

As for $\mathrm{CE}^I_{0-,\iota}$, (29) implies $\widehat{y}$ converges to the unique positive solution of $y(1+y) = \kappa p_I(1+p_I)$, which is $(1/2)(\sqrt{1+4\kappa p_I(1+p_I)} - 1)$ and enforces $(\kappa p_I(1+p_I) + y^2)/(1+2y) = y$. Therefore,

$$\frac{\kappa p_I(1+p_I) + \widehat{y}^2}{\kappa(1+p_I)(1+2\widehat{y})} \to \frac{\sqrt{1+4\kappa p_I(1+p_I)} - 1}{2\kappa(1+p_I)},$$

which gives

$$\lim_{\alpha_U\to\infty} \mathrm{CE}^I_{0-,\iota} = \alpha_I \frac{d}{2}\log\left(1+\frac{\sqrt{1+4\kappa p_I(1+p_I)}-1}{2\kappa(1+p_I)}\right).$$

Hence,

$$\begin{aligned}\lim_{\alpha_U\to\infty}\left(\mathrm{CE}^I_{0-,\iota}-\mathrm{CE}^I_{0-}\right) &= \frac{1}{2\kappa(1+p_I)}\left(\sqrt{1+4\kappa p_I(1+p_I)}-1-\frac{2\kappa p_I(1+p_I)}{1+\kappa p_I(1+p_I)}\right),\\ &=: \frac{1}{2\kappa(1+p_I)}\times f(\kappa p_I(1+p_I)).\end{aligned}$$

It is easy to see $f(x)>0$ for all $x>0$ and thus the statement follows. This finishes the proof of item (1).

For the second item, let us fix α_U and $\alpha_I\to 0$. This corresponds to both $\lambda\to 0$ and $\kappa\to 0$. As such, we again will write $\kappa=\alpha_I^2 p_N$ and appeal to (19) when analyzing $\widehat{y}$. For CE^I_{0-} one can see

$$\frac{(1-\lambda)^2\kappa p_I+(1+p_I)(\lambda+\kappa p_I)^2}{\kappa(1+\lambda p_I+\kappa p_I(1+p_I))^2}\to\frac{\alpha_U^2 p_I p_N+1+p_I}{\alpha_U^2 p_N}. \tag{75}$$

On the other hand, (19) implies $\widehat{y}/\alpha_I\to(1+p_I)/\alpha_U$. Therefore,

$$\frac{\kappa p_I(1+p_I)+\widehat{y}^2}{\kappa(1+p_I)(1+2\widehat{y})}\to\frac{\alpha_U^2 p_I p_N+1+p_I}{\alpha_U^2 p_N}.$$

The above limits give

$$\lim_{\alpha_I\to 0}\frac{\mathrm{CE}^I_{0-}}{\alpha_I}=\lim_{\alpha_I\to 0}\frac{\mathrm{CE}^I_{0-,\iota}}{\alpha_I}=\frac{1}{\alpha_U}\Pi'\mu_X-\frac{1}{\alpha_U^2}\Pi' P_X^{-1}\Pi+\frac{d}{2}\log\left(1+\frac{\alpha_U^2 p_I p_N+1+p_I}{\alpha_U^2 p_N}\right).$$

Now, the third order approximation for $\widehat{y}$ around $\alpha_I=0$ is

$$\widetilde{y}:=\frac{\alpha_I}{\alpha_U}(1+p_I)\left(1+\alpha_I p_I p_N(\alpha_U-\alpha_I p_I)\right)$$

and calculation shows $\limsup_{\alpha_I\to 0}\alpha_I^{-4}|\widehat{y}-\widetilde{y}|<\infty$. As such, in the limit

$$\lim_{\alpha_I\to 0}\frac{1}{\alpha_I^2}\left(\frac{\kappa p_I(1+p_I)+\widehat{y}^2}{\kappa(1+p_I)(1+2\widehat{y})}-\frac{(1-\lambda)^2\kappa p_I+(1+p_I)(\lambda+\kappa p_I)^2}{\kappa(1+\lambda p_I+\kappa p_I(1+p_I))^2}\right),$$

we can substitute $\widetilde{y}$ in for $\widehat{y}$ to obtain

$$\frac{(1+p_I)^2\left(1+p_I-\alpha_U^2 p_I p_N\right)}{\alpha_U^4 p_N}.$$

Using the above, limit (75), and the fact that $\lim_{x\to 0}\log(1+C(x)x)/x=C$ if $C(x)\to C$, we get

$$\lim_{\alpha_I\to 0}\frac{\mathrm{CE}^I_{0-,\iota}-\mathrm{CE}^I_{0-}}{\alpha_I^3}=\frac{d}{2}\frac{(1+p_I)^2(1+p_I-\alpha_U^2 p_I p_N)}{\alpha_U^2(\alpha_U^2 p_I p_N+1+p_I)}.$$

The latter completes the proof of the limiting arguments when $\alpha_I\to 0$.

We now send $\alpha_I\to\infty$. This causes both $\lambda\to 1$ and $\kappa\to\infty$ so we will (c.f. (28)) plug in $\kappa=\alpha_I^2 p_N$. Doing this, one can verify

$$\alpha_I^2\times\frac{(1-\lambda)^2\kappa p_I+(1+p_I)(\lambda+\kappa p_I)^2}{\kappa(1+\lambda p_I+\kappa p_I(1+p_I))^2}\to\frac{1}{p_N(1+p_I)}.$$

Since, $\alpha_I\log(1+C/\alpha_I^2)\to 0$ as $\alpha_I\to\infty$, we get that $\lim_{\alpha_I\to\infty}\mathrm{CE}^I_{0-}=\Pi'\mu_X$. As for $\mathrm{CE}^I_{0-,\iota}$, appealing to (19) we see that, by setting $\widehat{y}=\alpha_I\widehat{z}$, dividing by α_I^2 and taking the limit that $\widehat{z}$ converges to a solution of

$$0=z^2\left(1-\frac{\alpha_U z}{1+p_I}\right)+p_I p_N\left(\alpha_U z+1\right).$$

Furthermore, the exact same arguments as in the proof of Proposition 2.7 show there is a unique positive solution. Given that $\widehat{y}/\alpha_I \to \widehat{z}$ we find

$$\alpha_I \times \frac{\kappa p_I(1+p_I)+\widehat{y}^2}{\kappa(1+p_I)(1+2\widehat{y})} \to \frac{p_I p_N(1+p_I)+\widehat{z}^2}{2p_N(1+p_I)\widehat{z}}.$$

This gives that

$$\lim_{\alpha_I\to\infty} \mathrm{CE}^I_{0-,\iota} = \Pi'\mu_X + \frac{d\left(p_I p_N(1+p_I)+\widehat{z}^2\right)}{4p_N(1+p_I)\widehat{z}}$$

since $\alpha_I \log(1+C/\alpha_I) \to C$ as $\alpha_I \to \infty$. The above limits finish the proof of the second item.

Lastly, for item (3), we have from (28) that $\alpha_U = \alpha_I$ yields $\lambda = 1/2$. As such, (36) implies $\mathrm{CE}^I_{0-,\iota} \geq \mathrm{CE}^I_{0-}$ is equivalent to

$$f(\widehat{y}) := \frac{\kappa p_I(1+p_I)+\widehat{y}^2}{(1+2\widehat{y})} \geq \frac{(1+p_I)\left(\kappa p_I + (1+p_I)(1+2\kappa p_I)^2\right)}{(2+p_I+2\kappa p_I(1+p_I))^2} =: k,$$

where $\widehat{y}$ solves (29) with $\lambda = 1/2$. Note that

$$\dot{f}(y) = \frac{2(y(1+y)-\kappa p_I(1+p_I))}{(1+2y)^2},$$

so that f is minimized $(0,\infty)$ at $y_0 = (1/2)\left(\sqrt{1+4\kappa p_I(1+p_I)}-1\right)$, and this value enforces $f(y_0) = y_0$. Therefore, $f(\widehat{y}) \geq f(y_0) = y_0$ and hence if

$$y_0 \geq k \iff \frac{1}{2}\left(\sqrt{1+4\kappa p_I(1+p_I)}-1\right) \geq \frac{(1+p_I)\left(\kappa p_I + (1+p_I)(1+2\kappa p_I)^2\right)}{(2+p_I+2\kappa p_I(1+p_I))^2}, \tag{76}$$

then $\mathrm{CE}^I_{0-,\iota} \geq \mathrm{CE}^I_{0-}$. Otherwise, $f(y_0) < k$ is strictly less than the right side above, and denote by $\check{y}$ the unique $y > y_0$ such that $f(\check{y}) = k$. As (29) implies

$$\widehat{y}(1+\widehat{y}) - \kappa p_I(1+p_I) = \frac{(1+p_I)\left((1+\widehat{y})^2+\kappa p_I\right)}{1+\widehat{y}} > 0,$$

$\dot{f}(\widehat{y}) > 0$ and hence $\widehat{y} > y_0$. Therefore, $\mathrm{CE}^I_{0-,\iota} \geq \mathrm{CE}^I_{0-}$ is equivalent to $\widehat{y} \geq \check{y}$, which as shown in the proof of Proposition 2.7, is equivalent to $g(\check{y}) \geq 0$ for g defined by the cubic function in (29). However, it can easily be checked by any software tool that either (76) holds, or $g(\check{y}) \geq 0$. This gives the result.

□

Proof of Proposition 4.7. We will start with the price taking case and use the notation from (28). From Lemma C.1 and using (38) we find

$$\begin{aligned}
\gamma_I \mathrm{CE}^I_0(G, Z_N) &= \psi'_{I,0}\mu_X + \frac{p_I(\kappa p_I+\lambda)}{(1+p_I)\left(\frac{1-\lambda}{1+p_I}+\kappa p_I+\lambda\right)}\psi'_{I,0}P_X^{-1/2}\bigg(\mathcal{E}_X + \frac{1}{\sqrt{p_I}}\mathcal{E}_I + \frac{\mathcal{E}_N}{p_I\alpha_I\sqrt{p_N}} \\
&\quad - \frac{1+\kappa p_I}{p_I(\kappa p_I+\lambda)}P_X^{-1/2}\widehat{\Pi}\bigg) + \frac{(1-\lambda)^2 p_I^2}{2(1+p_I)^3\left(\frac{1-\lambda}{1+p_I}+\kappa p_I+\lambda\right)^2}\bigg\|\mathcal{E}_X + \frac{1}{\sqrt{p_I}}\mathcal{E}_I \\
&\quad - \frac{(1+p_I)(\kappa p_I+\lambda)}{(1-\lambda)p_I\alpha_I\sqrt{p_N}}\mathcal{E}_N + \frac{(1+\kappa p_I)(1+p_I)}{(1-\lambda)p_I}P_X^{-1/2}\widehat{\Pi}\bigg\|^2.
\end{aligned}$$

It then follows that almost surely

$$\begin{aligned}
\lim_{p_I\to 0}\gamma_I \mathrm{CE}^I_0(G, Z_N) &= \psi'_{I,0}\mu_X + \psi'_{I,0}P_X^{-1/2}\left(\frac{\lambda}{\alpha_I\sqrt{p_N}}\mathcal{E}_N - P_X^{-1/2}\widehat{\Pi}\right) \\
&\quad + \frac{1}{2}\left\|\frac{\lambda}{\alpha_I\sqrt{p_N}}\mathcal{E}_N - P_X^{-1/2}\widehat{\Pi}\right\|^2.
\end{aligned}$$

Substituting back in for Z_N in (38), and using (69) gives

$$\psi_{I,0}'\mu_X - \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0} + \psi_{I,0}'P_X^{-1}\left(\frac{\lambda_I}{\alpha_I}Z_N - (1-\lambda)\psi_{U,0} - \lambda\psi_{I,0}\right) + \frac{1}{2}\left\|P_X^{-1/2}\left(\frac{\lambda}{\alpha_I}Z_N - (1-\lambda)\psi_{U,0} - \lambda\psi_{I,0}\right)\right\|^2 + \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0},$$

and hence the result. We next consider the price impact case. Here

$$\gamma_I \mathrm{CE}_{0,\iota}^I(G, Z_N) = \psi_{I,0}'\mu_X + \psi_{I,0}'P_X^{-1/2}\Bigg(\frac{\widehat{y}}{1+p_I}\frac{1}{\alpha_I\sqrt{p_N}}\mathcal{E}_N - \frac{(\widehat{y}+1)^2+\kappa p_I}{(1-\lambda)(\widehat{y}+1)^2}P_X^{-1/2}\widehat{\Pi} - \frac{\kappa p_I\widehat{y}}{(\widehat{y}+1)^2}P_X^{-1/2}\psi_{I,0}\Bigg) + \frac{1}{2(1+p_I)(1+2\widehat{y})}\Bigg\|p_I\mathcal{E}_X + \sqrt{p_I}\mathcal{E}_I - \frac{\widehat{y}}{\alpha_I\sqrt{p_N}}\mathcal{E}_N + \frac{(1+p_I)((\widehat{y}+1)^2+\kappa p_I)}{(1-\lambda)(\widehat{y}+1)^2}P_X^{-1/2}\widehat{\Pi} + \frac{\widehat{y}\left(\frac{1}{1+p_I}(\widehat{y}+1)^2+\kappa p_I\right)(1+p_I)}{(\widehat{y}+1)^2}P_X^{-1/2}\psi_{I,0}\Bigg\|^2.$$

(29) implies $\widehat{y} \to \lambda/(1-\lambda)$ as $p_I \to 0$. It then follows that almost surely

$$\lim_{p_I\to 0}\gamma_I \mathrm{CE}_{0,\iota}^I(G, Z_N) = \psi_{I,0}'\mu_X + \psi_{I,0}'P_X^{-1/2}\left(\frac{\lambda}{(1-\lambda)\alpha_I\sqrt{p_N}}\mathcal{E}_N - \frac{1}{1-\lambda}P_X^{-1/2}\widehat{\Pi}\right) + \frac{1}{2(1-\lambda^2)}\left\|\frac{\lambda}{\alpha_I\sqrt{p_N}}\mathcal{E}_N - P_X^{-1/2}\widehat{\Pi} - \lambda P_X^{-1/2}\psi_{I,0}\right\|^2.$$

Substituting back in Z_N and plugging in for $\widehat{\Pi}$, the right side is

$$\psi_{I,0}'\mu_X - \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0} + \psi_{I,0}'P_X^{-1}\left(\frac{\lambda}{(1-\lambda)\alpha_I}Z_N - \psi_{U,0} - \frac{\lambda}{(1-\lambda)}\psi_{I,0}\right) + \frac{1}{2(1-\lambda^2)}\left\|P_X^{-1/2}\left(\frac{\lambda}{\alpha_I}Z_N - (1-\lambda)\psi_{U,0} - 2\lambda\psi_{I,0}\right)\right\|^2 + \frac{1}{2}\psi_{I,0}'P_X^{-1}\psi_{I,0}.$$

The result follows by collecting terms. We next turn to U where using (67) we obtain $P_X^{1/2}(H - \mu_X) = \mathcal{E}_X + 1/\sqrt{p_I}\mathcal{E}_I + 1/(p_I\alpha_I\sqrt{p_N})\mathcal{E}_N$. Therefore, from Lemma C.1

$$\gamma_U \mathrm{CE}_0^U(H) = \psi_{U,0}'\mu_X + \frac{(\kappa p_I + \lambda)}{(1+p_I)\left((1-\lambda)\frac{1}{1+p_I} + \kappa p_I + \lambda\right)}\psi_{U,0}'P_X^{-1/2}\Bigg(p_I\mathcal{E}_X + \sqrt{p_I}\mathcal{E}_I + \frac{1}{\alpha_I\sqrt{p_N}}\mathcal{E}_N - \frac{1+\kappa p_I}{\kappa p_I+\lambda}P_X^{-1/2}\widehat{\Pi}\Bigg) + \frac{(1+\kappa p_I)\lambda^2}{2(1+p_I)^3\left(\frac{1}{1+p_I}+\kappa p_I\right)\left(\frac{1-\lambda}{1+p_I}+\kappa p_I+\lambda\right)^2}\Bigg\|p_I\mathcal{E}_X + \sqrt{p_I}\mathcal{E}_I + \frac{1}{\alpha_I\sqrt{p_N}}\mathcal{E}_N - \frac{1+(1+p_I)\kappa p_I}{\lambda}P_X^{-1/2}\widehat{\Pi}\Bigg\|^2.$$

This gives the almost sure limit

$$\lim_{p_I\to 0}\gamma_U \mathrm{CE}_0^U(H) = \psi_{U,0}'\mu_X + \psi_{U,0}'P_X^{-1/2}\left(\frac{\lambda}{\alpha_I\sqrt{p_N}}\mathcal{E}_N - P_X^{-1/2}\widehat{\Pi}\right) + \frac{1}{2}\left\|\frac{\lambda}{\alpha_I\sqrt{p_N}}\mathcal{E}_N - P_X^{-1/2}\widehat{\Pi}\right\|^2.$$

Substituting back in Z_N and plugging in for $\widehat{\Pi}$, the right side is

$$\psi'_{U,0}\mu_X - \frac{1}{2}\psi'_{U,0}P_X^{-1}\psi_{U,0} + \psi'_{U,0}P_X^{-1}\left(\frac{\lambda}{\alpha_I}Z_N - (1-\lambda)\psi_{U,0} - \lambda\psi_{I,0}\right) + \frac{1}{2}\left\|P_X^{-1/2}\left(\frac{\lambda}{\alpha_I}Z_N - (1-\lambda)\psi_{U,0} - \lambda\psi_{I,0}\right)\right\|^2 + \frac{1}{2}\psi'_{U,0}P_X^{-1}\psi_{U,0},$$

from which the result can be deduced. Lastly we consider the price impact case where again using (67) we obtain $P_X^{1/2}(H_\iota - \mu_X) = \mathcal{E}_X + 1/\sqrt{p_I}\mathcal{E}_I + (\widehat{y}+1)/(p_I\alpha_I\sqrt{p_N})\mathcal{E}_N$. Therefore, from Lemma C.2

$$\begin{aligned}\gamma_U \mathrm{CE}^U_{0,\iota}(H_\iota) &= \psi'_{U,0}\mu_X + \psi'_{U,0}P_X^{-1/2}\Bigg(\frac{\widehat{y}}{(1+p_I)(1+2\widehat{y})}\left(p_I\mathcal{E}_X + \sqrt{p_I}\mathcal{E}_I + \frac{\widehat{y}+1}{\alpha_I\sqrt{p_N}}\mathcal{E}_N\right)\\
&+ \frac{\widehat{y}\lambda(\kappa p_I + (1+\widehat{y})^2)}{(1-\lambda)(1+2\widehat{y})(\widehat{y}+1)^2}P_X^{-1/2}\psi_{I,0} - \frac{\kappa p_I + (1+\widehat{y})^2}{(1-\lambda)(1+2\widehat{y})(1+\widehat{y})}P_X^{-1/2}\widehat{\Pi}\Bigg)\\
&+ \frac{1}{2}\frac{\lambda^2(\kappa p_I + (1+\widehat{y})^2)}{(1-\lambda)^2(1+p_I)(1+2\widehat{y})^2\left(\kappa p_I + \frac{1}{1+p_I}(1+\widehat{y})^2\right)}\Bigg\|p_I\mathcal{E}_X + \sqrt{p_I}\mathcal{E}_I + \frac{\widehat{y}+1}{\alpha_I\sqrt{p_N}}\mathcal{E}_N\\
&+ \frac{\left(\kappa p_I + \frac{1}{1+p_I}(1+\widehat{y})^2\right)\widehat{y}(1+p_I)}{(1+\widehat{y})^2}P_X^{-1/2}\psi_{I,0} - \frac{\left(\kappa p_I + \frac{1}{1+p_I}(1+\widehat{y})^2\right)(1+p_I)}{(1+\widehat{y})\lambda}P_X^{-1/2}\widehat{\Pi}\Bigg\|^2.\end{aligned}$$

Using $\widehat{y} \to \lambda/(1-\lambda)$ in the limit, we obtain the almost sure limit

$$\lim_{p_I\to 0}\gamma_U \mathrm{CE}^U_{0,\iota}(H_\iota) = \psi'_{U,0}\mu_X + \frac{1}{1-\lambda^2}\psi'_{U,0}P_X^{-1/2}\left(\frac{\lambda}{\alpha_I\sqrt{p_N}}\mathcal{E}_N + \lambda^2 P_X^{-1/2}\psi_{I,0} - P_X^{-1/2}\widehat{\Pi}\right) + \frac{1}{2(1-\lambda^2)^2}\left\|\frac{\lambda}{\alpha_I\sqrt{p_N}}\mathcal{E}_N + \lambda^2 P_X^{-1/2}\psi_{I,0} - P_X^{-1/2}\widehat{\Pi}\right\|^2.$$

Substituting back in Z_N and plugging in for $\widehat{\Pi}$, the right side is

$$\psi'_{U,0}\mu_X \pm \frac{1}{2}\psi'_{U,0}P_X^{-1}\psi_{U,0} + \frac{1}{\lambda_U(2-\lambda_U)}\psi'_{U,0}P_X^{-1}\left(\frac{\lambda_I}{\alpha_I}Z_N - \lambda_U\lambda_I\psi_{I,0} - \lambda_U\psi_{U,0}\right) + \frac{1}{2\lambda_U^2(2-\lambda_U)^2}\left\|P_X^{-1/2}\left(\frac{\lambda_I}{\alpha_I}Z_N - \lambda_U\lambda_I\psi_{I,0} - \lambda_U\psi_{U,0}\right)\right\|^2,$$

from which the result follows. □

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