OLS limit theory for drifting sequences of parameters on the explosive side of unity[1]

Tassos Magdalinos — University of Southampton

Katerina Petrova — NY Fed, Universitat Pompeu Fabra

9 July 2024

[1] The views expressed here are our own and do not necessarily reflect those of the Federal Reserve Bank of New York or any other part of the Federal Reserve System.

Introduction

- The paper contributes to the literature on uniform inference with AR(1) processes
- Drifting sequences of autoregressive parameters have been used to establish uniform procedures over AR parameter space $[-1+\delta, 1]$ for some $\delta > 0$; Mikusheva (2007).
- Andrews and Guggenberger (2014) considers certain distributional aspects of the innovation sequence of an autoregression as an infinite dimensional nuisance parameter

$$\Theta = [-1+\delta, 1] \times \{F : F \text{ d.f. of } u_t \text{ satisfying } \ldots\}$$

- (u_t) stationary (possibly conditionally heteroskedastic) martingale difference
- Andrews and Guggenberger (2012): OLS limit theory along drifting sequences $(\theta_n)_{n \in \mathbb{N}} \subset \Theta$
- $\theta_n = (\rho_n, F_n) \quad F_n(x) = \mathbb{P}(u_{n,t} \leq x)$

- Magdalinos and Petrova (2023, MP2023): uniform **IV** procedure for autoregression and predictive regression over $[-M, M]$ for some $M > 0$
- We would like to extend the above parameter space to include the innovation sequence as an infinite dimensional nuisance parameter

$$\Theta = [-M, M] \times \{(F_t) : (F_t) \text{ d.f. of } (u_t) \text{ satisfying ...}\}$$

1. Autoregression: (u_t) martingale difference; $F_t = F$ under conditional heteroskedasticity
2. Predictive regression: $u_t = \sum_{j=0}^{\infty} c_j e_{t-j}$ with (e_t) as in 1.

- MP2023: **IV** limit theory along innovations

$$u_{n,t} = \sum_{j=0}^{\infty} c_{n,j} e_{n,t-j}, \quad \rho_n \to (-\infty, \infty) \tag{1}$$

- This (sister) paper: **OLS** limit theory along (1) when

$$\rho_n \to (-\infty, -1] \cup [1, \infty)$$

Plan for the Talk

- Rôle of asymptotics along drifting sequences of parameters for uniform inference
- LLN for $n^{-1}\sum_{t=1}^{n} u_{n,t}^2$
- Mildly explosive case: $|\rho_n| \longrightarrow 1$ from the explosive side: martingale *CLT* carries over well to arrays
- Explosive case: $|\rho_n| \rightarrow |\rho| > 1$: no CLT; martingale *convergence* theorem does not generalise to arrays
- Limit distribution of the OLS estimator and OLS t-statistic

Rôle of asymptotics along drifting sequences

- Consider a CI $I_n(c_\alpha)$ for ρ: $S_n(\rho)$ statistic, c_α cv.
- Coverage of the CI:

$$\pi_n(\alpha) = \inf_{\theta\in\Theta} \mathbb{P}_\theta\left(|S_n(\rho)| \leq c_\alpha\right)$$

- The d.f. of $S_n(\rho)$ depends on $\theta = (\rho, F) \in \Theta$
- Correct asymptotic coverage (uniformly over Θ):

$$\liminf_{n\to\infty} \pi_n(\alpha) \geq 1-\alpha. \tag{2}$$

- By the definition of infimum, we may select **some** $(\theta_n)_{n\in\mathbb{N}}$ in Θ s.t.

$$\liminf_{n\to\infty} \pi_n(\alpha) \geq \liminf_{n\to\infty} \mathbb{P}_{\theta_n}\left(|S_n(\rho_n)| \leq c_\alpha\right)$$

- Sufficient condition for (2):

$$\liminf_{n\to\infty} \mathbb{P}_{\theta_n}\left(|S_n(\rho_n)| \leq c_\alpha\right) \geq 1-\alpha \quad \textbf{for any} \quad (\theta_n)_{n\in\mathbb{N}} \subseteq \Theta \tag{3}$$

- (3) requires computing $\lim_{n\to\infty} \mathbb{P}_{\theta_n}\left(|S_n(\rho_n)| \leq x\right)$

Rôle of subsequential arguments (BW type)

- Convergence in distribution along any $(\theta_n)_{n\in\mathbb{N}} \subseteq \Theta$:

$$\lim_{n\to\infty} \mathbb{P}_{\theta_n}\left(S_n\left(\rho_n\right) \leq x\right) = \varphi\left(x\right), \quad x \in \mathcal{C}_\varphi \qquad (4)$$

- Equivalently: $\forall\left(m_n\right) \subseteq \mathbb{N} \;\; \exists\left(k_n\right) \subseteq \left(m_n\right)$

$$\lim_{n\to\infty} \mathbb{P}_{\theta_{k_n}}\left(S_{k_n}\left(\rho_{k_n}\right) \leq x\right) = \varphi\left(x\right), \quad x \in \mathcal{C}_\varphi \qquad (5)$$

- (5) allows us to employ the BW theorem
- Example: linear process weights:

$$c_0 = 1, \;\; \sum_{j=0}^{\infty} \left|c_j\right| \leq M < \infty, \;\; \left|\sum_{j=0}^{\infty} c_j\right| \geq \delta > 0 \qquad (6)$$

- Drifting sequence $\left(c_{n,j}\right)_{n\in\mathbb{N}}$ from (6) satisfies $\sup_{n\in\mathbb{N}} \sum_{j=0}^{\infty} \left|c_{n,j}\right| < \infty$.
- BW implies that $\exists\left(k_n\right)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ s.t.

$$c_{k_n,0} \to 1, \;\; \sum_{j=0}^{\infty} \left|c_{k_n,j}\right| \to C < \infty, \;\; \sum_{j=0}^{\infty} c_{n,j} \to C\left(1\right) \neq 0$$

Assumptions on AR root

$$\begin{aligned} x_{n,t} &= \rho_n x_{n,t-1} + u_{n,t} \\ u_{n,t} &= \sum_{j=0}^{\infty} c_{n,j} e_{n,t-j} \end{aligned}$$

Assumption: $\rho_n \to \rho \in (-\infty, -1] \cup [1, \infty)$ and $n\left(|\rho_n| - 1\right) \to \infty$

- Mildly explosive root: $\rho \in \{-1, 1\}$
- Explosive root: $\rho \in (-\infty, -1) \cup (1, \infty)$
- Oscillating roots: $\rho \in (-\infty, -1]$

Assumptions on innovations

$u_{n,t} = \sum_{j=0}^{\infty} c_{n,j} e_{n,t-j}$

1. $\sup_{n\geq 1} \sum_{j=0}^{\infty} j^{\delta} |c_{n,j}| < \infty$ *some* $\delta > 0$

$$C(\rho) := \lim_{n\to\infty} \sum_{j=0}^{\infty} \rho^{-j} c_{n,j} \neq 0, \quad \lim_{n\to\infty} \sum_{j=0}^{\infty} c_{n,j}^2 > 0$$

2. $(e_{n,t}, \mathcal{F}_{n,t})_{t\in\mathbb{Z}}$ *is a martingale difference array such that*

$$\liminf_{n\to\infty} \liminf_{t\to\infty} \mathbb{E}_{\mathcal{F}_{n,t-1}} |e_{n,t}| > 0 \ \ a.s.$$

$(e_{n,t}^2)_{n\in\mathbb{N},t\in\mathbb{Z}}$ *is UI,* $\sigma_n^2 := \mathbb{E}(\sigma_{n,t}^2) \to \sigma^2 > 0$ *and*
$\sigma_{n,t}^2 := \mathbb{E}_{\mathcal{F}_{n,t-1}}(e_{n,t}^2)$ *satisfies* **one** *of:*

1. $\sigma_{n,t}^2 = \sigma_n^2$ a.s. *for all* t.
2. $(e_{n,t}, \sigma_{n,t})_{t\in\mathbb{Z}}$ *is strictly stationary with* $\sigma_{n,t}^2 > 0$,

$$\limsup_{n\to\infty} \sup_{t\in\mathbb{N}} \sigma_{n,t}^2 < \infty \ a.s., \quad \left\{\sigma_{n,t}^2 : n \in \mathbb{N}\right\} \text{ is UI}$$

and $\exists b > 0$, $\exists (\psi_m)_{m\in\mathbb{N}}$ *satisfying* $\psi_m \to 0$ *such that*

$$\limsup_{n\to\infty} \left\| \mathbb{E}_{\mathcal{F}_{n,t-1-m}} \left(\sigma_{n,t}^2 - \sigma_n^2\right) \right\|_{L_1} \leq b\psi_m \quad \text{for all } t, m \geq 1.$$

Discussion

- $\sup_{n\geq 1}\sum_{j=0}^{\infty} j^{\delta}\left|c_{n,j}\right| < \infty$ *some* $\delta > 0$ needed to ensure

$$\sup_{n\geq 1}\sum_{j=m_n}^{\infty}\left|c_{n,j}\right| \to 0 \text{ when } m_n \to \infty \qquad (7)$$

- It is a tempting fallacy to think that (7) holds for $\delta = 0$
- Counterexample: $c_{n,j}^2 := \left(\phi_n^2 - 1\right)\phi_n^{2j}$ where $\phi_n \to 1$ with $n\left(\phi_n - 1\right) \to \infty$ satisfies $\sum_{j=0}^{\infty} c_{n,j}^2 \to 1$; take $m_n \to \infty$ and $m_n\left(\phi_n - 1\right) \to 0$

$$\sum_{j=0}^{m_n-1} c_{n,j}^2 \to 0 \text{ and } \sum_{j=m_n}^{\infty} c_{n,j}^2 \to 1.$$

- The condition

$$\limsup_{n\to\infty}\left\|\mathbb{E}_{\mathcal{F}_{n,t-1-m}}\left(\sigma_{n,t}^2 - \sigma_n^2\right)\right\|_{L_1} \leq b\psi_m$$

is very similar to a L_1-*mixingale array* of Andrews (1988)

Marcinkiewicz-Zygmund conditions

- Marcinkiewicz-Zygmund (1937); Lai and Wei (1983)
- A martingale difference sequence $(\eta_t, \mathcal{H}_t)_{t\in\mathbb{N}}$ is said to satisfy the *local MZ conditions* if

$$\liminf_{t\to\infty} \mathbb{E}\left(|\eta_t| \,\middle|\, \mathcal{H}_{t-1}\right) > 0 \; a.s. \tag{8}$$

$$\sup_{t\in\mathbb{N}} \mathbb{E}\left(\eta_t^2 \,\middle|\, \mathcal{H}_{t-1}\right) < \infty \; a.s. \tag{9}$$

- Lai and Wei (1983): if $(\eta_t, \mathcal{H}_t)_{t\in\mathbb{N}}$ satisfy (8) and (9), $\pi_j \neq 0$ for infinitely many j and $\sum_{j=1}^{\infty} \pi_j^2 < \infty$ then

$$\mathbb{P}\left(\sum_{j=1}^{\infty} \pi_j \eta_j = Y\right) = 0 \tag{10}$$

  for any r.v. Y that is $\mathcal{H}_t$-measurable for some $t \geq 1$.
- A series of the type in (10) appears in the denominator of the limit distribution of the OLS estimator in the explosive case

Infinite order ARCH

- Let $(\varepsilon_t, \mathcal{F}_t)_{t\in\mathbb{Z}}$ be a strictly stationary martingale difference satisfying $\mathbb{E}\varepsilon_1^2 < \infty$

$$\liminf_{t\to\infty} \mathbb{E}\left(|\varepsilon_t| | \mathcal{F}_{t-1}\right) \geq \delta > 0, \quad \sup_{t\in\mathbb{Z}} \sigma_t^2 < \infty \quad a.s. \tag{11}$$

  with conditional variance given by an ARCH(∞):

$$\sigma_t^2 = \omega + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-i}^2 \quad \sum_{i=1}^{\infty} \alpha_i \leq 1-\delta \quad \sum_{i=1}^{\infty} i^{\delta}\alpha_i \leq M \tag{12}$$

- Giraitis et al. (2000)): (12) includes all stationary GARCH(p,q) processes ε_t with $\mathbb{E}\varepsilon_1^2 < \infty$, (11), and ARCH$(\infty)$ representation satisfying $\sum_{i=1}^{\infty} \alpha_i \leq 1-\delta$
- With some effort, we may show the following:
- *For any sequence $(e_{n,t}, \mathcal{F}_{n,t})_{n,t\in\mathbb{Z}} \subseteq (\varepsilon_t, \mathcal{F}_t)_{t\in\mathbb{Z}}$ there exists a further subsequence $(e_{k_n,t}, \mathcal{F}_{k_n,t})_{n,t\in\mathbb{Z}}$ satisfying Assumption 2b.*

Assumptions on initial condition

(i) *When* $|\rho_n| \to 1$, $X_{n,0} = o_p\left[\left(\rho_n^2 - 1\right)^{-1/2}\right]$.

(ii) *When* $|\rho_n| \to |\rho| > 1$,

$$X_{n,0} \to_d X_0$$

where X_0 *is an* $\mathcal{F}_0$*-measurable random variable, where*

$$\mathcal{F}_0 := \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{n,0}\right)$$

Law of large numbers

- $u_{n,t} = \sum_{j=0}^{\infty} c_{n,j} e_{n,t-j}$, $\sigma_n^2 = \mathbb{E} e_{n,t}^2 \to \sigma^2$

Lemma

Let $c_{n,j}$ and $e_{n,t}$ satisfy Assumptions 1 and 2 hold and K be a non-negative bounded function on $[0,1]$ satisfying $K(0) = 1$. Then:

(i) $\left\| \frac{1}{n} \sum_{t=1}^{n} u_{n,t}^2 - \sigma^2 \lim_{n\to\infty} \sum_{j=0}^{\infty} c_{n,j}^2 \right\|_{L_1} \to 0.$

(ii) If, in addition, $\sup_{n\geq 1} \sup_{h\geq 1} \left| cov\left(e_{n,t}^2, e_{n,t+h}^2\right)\right| < \infty$ *holds, then*

$$\left\| \frac{1}{n} \sum_{h=1}^{M} K\left(\frac{h}{M}\right) \sum_{t=h+1}^{n} u_{n,t} u_{n,t-h} - \lim_{n\to\infty} \sum_{h=1}^{\infty} \gamma_{u_n}(h) \right\|_{L_1} \to 0$$

whenever $M \to \infty$ *and* $M/n^{1/2} \to 0$.

MG approximation (mildly explosive)

- Consider the stochastic sequences

$$\left[\mathbf{X}_n, \tilde{X}_n(\rho)\right] = \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} \left[u_{n,t}, C(\rho) e_{n,t}\right]$$

$$\left[\mathbf{Y}_n, \tilde{Y}_n(\rho)\right] = \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-(n-t+1)} \left[u_{n,t}, C(\rho) e_{n,t}\right]$$

Lemma

Let $\rho_n \to 1$ and Assumptions 1 and 2 hold. Then

$$\left\|\mathbf{X}_n - \tilde{X}_n(1)\right\|_{L_2} \to 0, \ \left\|\mathbf{Y}_n - \tilde{Y}_n(1)\right\|_{L_2} \to 0$$

and

$$\left[\mathbf{X}_n, \mathbf{Y}_n\right] \to_d \left[\tilde{X}(1), \tilde{Y}(1)\right]$$

where $\tilde{X}(1)$ and $\tilde{Y}(1)$ are independent $N\left(0, \sigma^2 C(1)^2\right)$ rv's.

Explosive case: MZ approximation

- $\rho_n \to \rho \in (1, \infty)$
- Let $(e_{n,t}, \mathcal{F}_{n,t})$ satisfy Assumptions 1 and 2.
- We would like to approximate $(e_{n,t}, \mathcal{F}_{n,t})$ along a subsequence by $(e_t, \mathcal{F}_t)$ that satisfies the local MZ conditions.
- We may take $e_t := \liminf_{n\to\infty} e_{n,t}$
- If $\mathcal{F}_{n,t}$ were increasing in n, $\mathcal{F}_{n,t} \uparrow \mathcal{F}_t := \sigma\left(\cup_{n=1}^{\infty} \mathcal{F}_{n,t}\right)$
- Levy's "upward" theorem would imply that

$$\begin{aligned}
\mathbb{E}\left(e_t \middle| \mathcal{F}_{t-1}\right) &= \lim_{m\to\infty} \mathbb{E}\left(e_t \middle| \mathcal{F}_{m,t-1}\right) = \lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}\left(e_{k_n,t} \middle| \mathcal{F}_{m,t-1}\right) \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}\left(\mathbb{E}\left(e_{k_n,t} \middle| \mathcal{F}_{k_n,t-1}\right) \middle| \mathcal{F}_{m,t-1}\right) \quad (k_n \geq m) \\
&= 0
\end{aligned}$$

- $\mathcal{F}_{n,t}$ does not have to be increasing in n

Explosive case: MZ approximation

- Construct an increasing sequence of σ-algebras from $\mathcal{F}_{n,t}$:

$$\mathcal{G}_{n,t} := \cap_{j=n}^{\infty} \mathcal{F}_{j,t} \quad \textit{and} \quad \mathcal{F}_t := \sigma\left(\cup_{n=1}^{\infty} \mathcal{G}_{n,t}\right) = \sigma\left(\liminf_{n\to\infty} \mathcal{F}_{n,t}\right)$$

- $e_{n,t}$ is not $\mathcal{G}_{n,t}$-measurable but is $\mathcal{F}_t$-measurable $\forall n \geq n_0$
- $e_t := \liminf_{n\to\infty} e_{n,t}$ is $\mathcal{F}_t$-measurable.
- Since $\mathcal{G}_{n,t} \subseteq \mathcal{F}_{n,t}$,

$$\mathbb{E}\left(e_{n,t} \middle| \mathcal{G}_{n,t-1}\right) = \mathbb{E}\left(\mathbb{E}\left(e_{n,t} \middle| \mathcal{F}_{n,t-1}\right) \middle| \mathcal{G}_{n,t-1}\right) = 0$$

- For (k_n) defined by $\lim_{n\to\infty} e_{k_n,t} := \liminf_{n\to\infty} e_{n,t}$,

$$\begin{aligned}\left\|\mathbb{E}\left(e_t \middle| \mathcal{F}_{t-1}\right)\right\|_{L_1} &\leq \left\|\mathbb{E}\left(e_t \middle| \mathcal{G}_{k_n,t-1}\right) - \mathbb{E}\left(e_t \middle| \mathcal{F}_{t-1}\right)\right\|_{L_1} + \left\|\mathbb{E}\left(e_t - e_{k_n,t} \middle| \mathcal{G}_{k_n,t-1}\right)\right\|_{L_1} \\ &\leq \left\|\mathbb{E}\left(e_t \middle| \mathcal{G}_{k_n,t-1}\right) - \mathbb{E}\left(e_t \middle| \mathcal{F}_{t-1}\right)\right\|_{L_1} + \left\| e_{k_n,t} - e_t\right\|_{L_1}\end{aligned}$$

- Taking $n \to \infty$, $\left\|\mathbb{E}\left(e_t \middle| \mathcal{F}_{t-1}\right)\right\|_{L_1} = 0$, i.e. $(e_t, \mathcal{F}_t)$ is a MG difference sequence.

- $\tilde{\mathbf{X}}_n := \mathbf{X}_n + \left(\rho^2 - 1\right)^{1/2} X_{n,0}$
- $e_t := \liminf_{n\to\infty} e_{n,t}, \quad \mathcal{F}_t := \sigma\left(\liminf_{n\to\infty} \mathcal{F}_{n,t}\right)$

Lemma

Let $\rho_n \to \rho \in (1, \infty)$ and Assumptions 1 and 2 hold. Then:
(i) The sequence $(e_t, \mathcal{F}_t)_{t\in\mathbb{Z}}$ is a martingale difference satisfying the local MZ conditions.
(ii) For every $(k_n) \subseteq \mathbb{N}$,

$$\tilde{\mathbf{X}}_{k_n} \to_d X_\infty := \left(\rho^2 - 1\right)^{-1/2} C(\rho) \sum_{i=1}^{\infty} \rho^{-i} e_i + G_0$$

where G_0 is a $\mathcal{F}_0$-measurable random variable, $\mathbb{P}(X_\infty = 0) = 0$ and $|\mathbf{Y}_n| / \left|\tilde{\mathbf{X}}_n\right| = O_p(1)$.

Oscillating processes

- $\rho_n \to \rho \in (-\infty, -1]$
- $x_t \longmapsto (-1)^t x_t$
- transforms an *oscillating* process into a *regular* process: $x_t^+ = (-1)^t x_t$ satisfies

$$x_t^+ = |\rho_n| x_{t-1}^+ + (-1)^t u_t$$

- Same asymptotic development with the $\rho \geq 1$ case with

$$[\mathbf{X}_n, \mathbf{Y}_n] = (\rho_n^2 - 1)^{1/2} \sum_{t=1}^n \left[\rho_n^{-t} u_{n,t}, \rho_n^{-(n-t+1)} u_{n,t}\right]$$

replaced by

$$\left[\hat{\mathbf{X}}_n, \hat{\mathbf{Y}}_n\right] = (\rho_n^2 - 1)^{1/2} \sum_{t=1}^n \left[|\rho_n|^{-t} (-1)^{-t} u_{n,t}, |\rho_n|^{-(n-t+1)} (-1)^{-t} u_{n,t}\right]$$

OLS limit distribution (mildly explosive)

Theorem

Let $|\rho_n| \to 1$, $n(|\rho_n| - 1) \to \infty$ and Assumptions 1 and 2 hold. The following limit theory applies to the OLS estimator $\hat{\rho}_n$ as $n \to \infty$:

$$\left(\rho_n^2 - 1\right)^{-1} |\rho_n|^n \left(\hat{\rho}_n - \rho_n\right) \to_d \mathcal{C}$$

where $\mathcal{C}$ denotes a standard Cauchy random variable.

- Direct generalisation of the Cauchy limit result Phillips and Magdalinos (2007) and subsequent papers to drifting sequences.

Explosive OLS limit distribution (general)

- $|\rho_n| \to |\rho| > 1$

$$\left(\rho_n^2 - 1\right)^{-1} |\rho_n|^n \left(\hat{\rho}_n - \rho_n\right) = \frac{\zeta_n}{\xi_n + \left(\rho^2 - 1\right)^{1/2} X_{n,0}} + o_p(1)$$

- $(\zeta_n, \xi_n) = (\mathbf{Y}_n, \mathbf{X}_n)$ *when* $\rho > 1$
- $(\zeta_n, \xi_n) = \left(-\hat{\mathbf{Y}}_n, \hat{\mathbf{X}}_n\right)$ *when* $\rho < -1$
- $\forall (k_n) \subseteq \mathbb{N},\ \ \xi_{k_n} + \left(\rho^2 - 1\right)^{1/2} X_{k_n,0} \to_d \xi_\infty \neq 0$ *a.s.*

$$\hat{\rho}_n - \rho_n = O_p\left(|\rho_n|^{-n}\right).$$

- $\mathbb{E}\zeta_n = \mathbb{E}\xi_n = 0,\ V(\zeta_n) = V(\xi_n)$ *and* $\lim_{n\to\infty} \mathbb{E}(\zeta_n \xi_n) = 0$

Explosive OLS limit distribution (under Gaussianity)

- *If* $\left(e_{n,t}\right)_{t\in\mathbb{Z}}$ *is Gaussian and*

$$\upsilon\left(\rho\right)=\lim_{n\to\infty}\sum_{j=0}^{\infty}\left(\sum_{t=1}^{\infty}\rho^{-t}c_{n,j+t}\right)^{2}$$

exists, then

$$\left(\rho_n^2-1\right)^{-1}\left|\rho_n\right|^n\left(\hat{\rho}_n-\rho_n\right)\to_d\frac{\zeta}{\xi+X_0\left\{C\left(\rho\right)^2/\left(\rho^2-1\right)+\upsilon\left(\rho\right)\right\}^{-1/2}}\qquad(13)$$

where ζ *and* ξ *are independent* $N\left(0,\sigma^2\right)$ *random variables*

- If $X_0=0$ *a.s. the right side of (13) follows a Cauchy distribution.*

t-statistic limit distribution

Theorem

Let $n(|\rho_n| - 1) \to \infty$ *and Assumptions 1 and 2 hold. The t-statistic satisfies* $T_n(\rho_n) \to_d N(0,1)$ *under each of the following conditions:*

(i) $|\rho_n| \to 1$

(ii) $|\rho_n| \to |\rho| > 1$ *and* $(e_{n,t})_{t\in\mathbb{Z}} = (u_{n,t})_{t\in\mathbb{Z}}$ *is independent Gaussian*