Network Disruption Under Incomplete Information

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Motivation

- Network participants rarely have complete information about the full architecture of the network.
- Common knowledge of network architecture is a typical assumption in the literature.
- What are the key nodes under agent uncertainty in network architecture?

Related Literature

- **Optimal Structural Intervention**
  - Ballester, Calvo, and Zenou (2006), Galeotti, Golub, and Goyal (2020), Sun, Zhao, and Zhou (2023)

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Basic Notations

- Adjacency representation of a graph:

$$\mathbf{g} = [g_{ij}] \in \{0,1\}^{n \times n}, \mathbf{g} = \mathbf{g}^T \text{and } g_{ii} = 0$$

- $N_i(\mathbf{g}) = \{j : g_{ij} = 1\}$ denotes the *neighborhood* of player i.
- $d_i(\mathbf{g}) = |N_i(\mathbf{g})|$ denotes the *degree* of player i.

Timing of Events

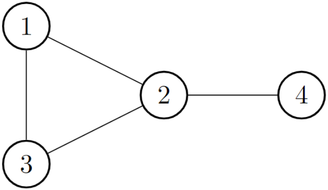
- Nature moves first and picks an undirected and unweighted network on n vertices from an ex-ante probability distribution over graphs on n vertices.
- Each player realizes their connections (neighbors) but fails to see beyond that.
- A fully informed and strategic disruptor structurally disrupts the network to minimize the aggregate activity.
- Agents exert efforts simultaneously to maximize the interim linear quadratic payoffs.

Disruption

- The disruptor has complete information about the realized network and the effort game played by agents.
- **Assumption 1:** Disruption is not anticipated by the players (this is relaxed in the paper)
- Two kinds of Disruption strategies:
  1. **Node removal**
  2. Links removal
- Incomplete info. implies that disruptor's actions are only observed locally.

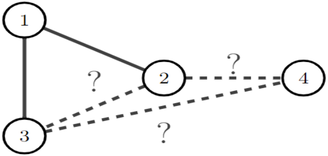
Graph Realization

Suppose the following graph involving 4 players is realized:

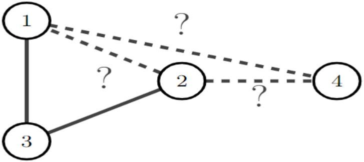


$$\Longrightarrow \mathbf{g} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

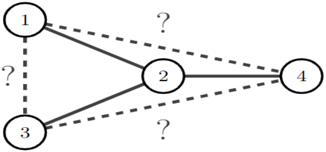
Realization of each player



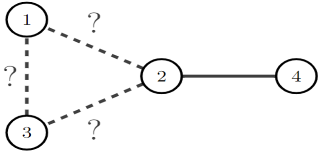
Player 1



Player 3

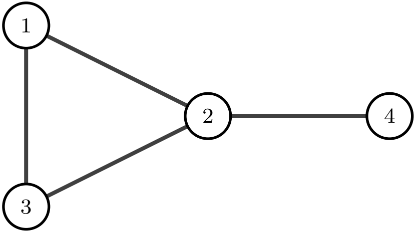


Player 2

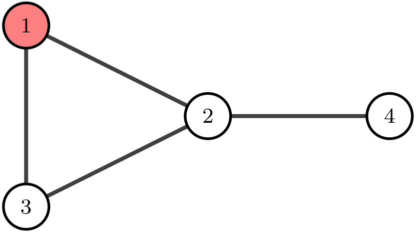


Player 4

Disruption



Disruption (Node Removal)



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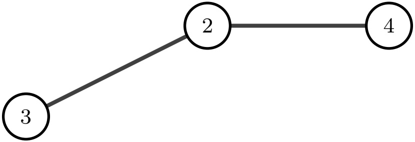


Figure 1: Player 1 is removed

Incomplete Information Game (CJST(2023))

- Types of each player correspond to the identity of their neighbors in the realized graph.
- G_i is the set of types for each player $i \in N$, where
$$G_i = \{\mathbf{g}_i = (g_{i1}, g_{i2}, ..., g_{in})_i \in \{0,1\}^n : g_{ii} = 0\}$$
- Players update their beliefs:

$$p(\mathbf{g}_j|\mathbf{g}_i) = \frac{p(\mathbf{g}_i, \mathbf{g}_j)}{p(\mathbf{g}_i)} = \frac{\sum_{\mathbf{g}\in G} p(\mathbf{g})\mathbb{I}\{\mathbf{g}_i = \mathbf{g}|_i \wedge \mathbf{g}_j = \mathbf{g}|_j\}}{\sum_{\mathbf{g}\in G} p(\mathbf{g})\mathbb{I}\{\mathbf{g}_i = \mathbf{g}|_i\}}$$

Incomplete Information Game (CJST(2023))

- $a_i(\mathbf{g}_i^{t_i}) \in A \equiv [0, \infty)$ is the type dependent effort exerted by each player.
- The interim payoff structure:

$$u_i(a_i(\mathbf{g}_i^{t_i}); \mathbf{a}_i(\mathbf{g}_i^{-t_i}), \mathbf{a}_{-i}) = a_i(\mathbf{g}_i^{t_i}) - \frac{1}{2}a_i(\mathbf{g}_i^{t_i})^2 + \lambda a_i(\mathbf{g}_i^{t_i}) \sum_{j=1}^{n} g_{ij}^{t_i} \sum_{\mathbf{g}_j \in G_j} p(\mathbf{g}_j | \mathbf{g}_i^{t_i}) a_j(\mathbf{g}_j)$$

where $\lambda > 0$.

Incomplete Information Game (CJST(2023))

- We use the **Bayes Nash** equilibrium (BNE) notion.
- The pure strategy profile $\sigma^* = (\sigma_i^*,\ \sigma_{-i}^*)$ where $\sigma_i = (a_i^*(\mathbf{g}_i^1), ..., a_i^*(\mathbf{g}_i^\gamma))$ is a BNE if for all $i \in N$ and $\mathbf{g}_i^{t_i} \in G_i$,

$$a_i^*(\mathbf{g}_i^{t_i}) = arg \max_{a_i(\mathbf{g}_i^{t_i})} u_i(a_i(\mathbf{g}_i^{t_i}), \mathbf{a}_i^*(\mathbf{g}_i^{-t_i}), \sigma_{-i}^*)$$

- Existence and uniqueness of BNE if $\lambda \in [0, \frac{1}{n-1})$

Key Node

- For a realized graph $\tilde{\mathbf{g}}$, when player k is removed, the best response of a player $i \in N_k$:

$$\tilde{a}_i\left(\tilde{\mathbf{g}}_i\right) = 1 + \lambda \sum_{j \neq k} \tilde{g}_{ij} \sum_{\{\mathbf{g_j} \in G_j : g_{jk} = 1\}} p\left(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i\right) \tilde{a}_j\left(\mathbf{g}_j\right) + \lambda \sum_{j \neq k} \tilde{g}_{ij} \sum_{\{\mathbf{g_j} \in G_j : g_{jk} = 0\}} p\left(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i\right) a_j^*\left(\mathbf{g}_j\right)$$

- Key Node:

$$\arg\max_{k \in N} \Psi\left(k\right) := \Sigma - \Sigma_{-k}$$

where,

$$\Sigma = \sum_{i \in N} a_i^*\left(\tilde{\mathbf{g}}_i\right) \quad \text{and} \quad \Sigma_{-k} = \sum_{i \in N_k} \tilde{a}_i\left(\tilde{\mathbf{g}}_i\right) + \sum_{\substack{i \notin N_k \\ i \neq k}} a_i^*\left(\tilde{\mathbf{g}}_i\right)$$

Key Node

Theorem 1

For any admissible probability distribution, and for any realized network $\tilde{\mathbf{g}} \in G$, the key node k^* is the one for whom

$$\Psi(k^*) := a^*_{k^*}(\tilde{\mathbf{g}}_{k^*}) + \lambda \sum_{i \in N_{k^*}} \tilde{\Psi}_{k^*}(\tilde{\mathbf{g}}_i)$$

is maximal, i.e. $\Psi(k^*) = \max_{k \in N} \Psi(k)$, with

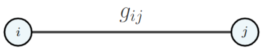
$$\tilde{\Psi}_{k^*}(\tilde{\mathbf{g}}_i) = \sum_{s=0}^{\infty} \lambda^s \psi^{(s)}_{i,t_i} \quad \forall\, i \in N$$

where,

$$\psi^{(1)}_{i,t_i} = \sum_{j=1}^{n} \sum_{\mathbf{g}_j \in G^2_j} g_{ij} p\left(\mathbf{g}_j | \mathbf{g}_i^{t_i}\right) \Phi_{k^*}(\mathbf{g}_j) \quad \text{and} \quad \Phi_{k^*}(\mathbf{g}_j) = \mathbb{E}_j\left[a^*_{k^*} | \mathbf{g}_j\right]$$

Expected Complementarity

$$\psi_{i,t_i}^{(2)} = \sum_{j,l=1} \sum_{\mathbf{g}_j \in G_j^2} \sum_{\mathbf{g}_l \in G_l^2} g_{ij} g_{jl} p(\mathbf{g}_j | \mathbf{g}_i^{t_i}) p(\mathbf{g}_s | \mathbf{g}_j) \Phi_{k^*} (\mathbf{g}_l)$$



- Certain about the links between i and j, i.e. g_{ij}.

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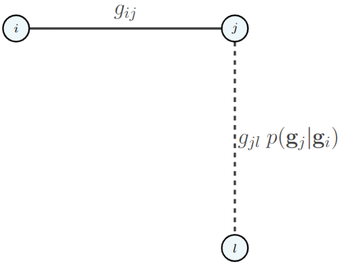

- Certain about the links between i and j, i.e. g_{ij}.
- Given the link with j, the possible walks from i to l through j is given by $g_{jl} p(\mathbf{g}_j|\mathbf{g}_i)$.

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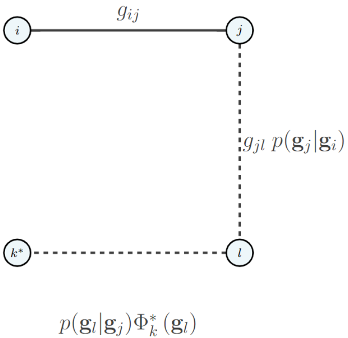

- Certain about the links between i and j, i.e. g_{ij}.
- Given the link with j, the possible walks from i to l through j is given by $g_{jl} p(\mathbf{g}_j|\mathbf{g}_i)$.
- The complementarity that i expects l to extract from k^*, i.e. $p(\mathbf{g}_l|\mathbf{g}_j)\Phi_k^*(\mathbf{g}_l)$

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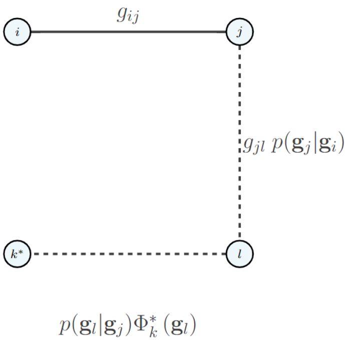

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- Given the link with j, the possible walks from i to l through j is given by $g_{jl} p(\mathbf{g}_j|\mathbf{g}_i)$.
- The complementarity that i expects l to extract from k^*, i.e. $p(\mathbf{g}_l|\mathbf{g}_j)\Phi_k^*(\mathbf{g}_l)$
- $g_{ij} g_{jl} p(\mathbf{g}_j|\mathbf{g}_i^{t_i}) p(\mathbf{g}_l|\mathbf{g}_j)\Phi_k^*(\mathbf{g}_l)$ is the complementarity that player i expects to extract from player k^* through this walk.

Implication

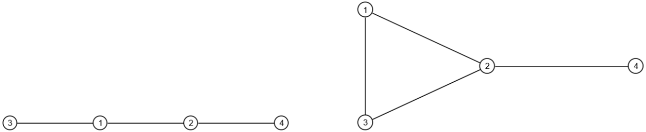


Figure 2: $\mathbf{g}^1$ (left) and $\mathbf{g}^2$ (right)

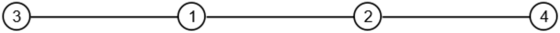
Suppose that the ex-ante distribution p satisfies:

$$p(\mathbf{g}) = \begin{cases} 0.3 & \text{if } \mathbf{g} = \mathbf{g}^1 \\ 0.7 & \text{if } \mathbf{g} = \mathbf{g}^2 \\ 0 & \text{otherwise} \end{cases}$$

.

Implication

Suppose the graph $\mathbf{g}^1$ has been realized.



- Player 1 and Player 2 have the highest intercentrality metric.
- Player 2 is the key player under incomplete information.

Uniform Case

Lemma 1

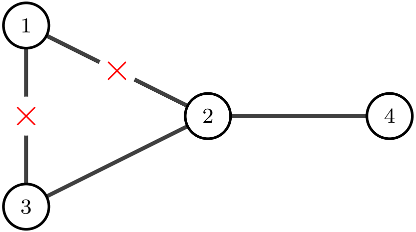
When the underlying probability distribution is uniform For any graph realization with $n \geq 3$ an individual with the highest degree is the key node.

- The expected complementarity that each player expects to extract through their neighbors is the same for everyone.

Conclusion

- Optimal disruption strategies under incomplete information on graph realizations.
- The intercentrality metric is no longer sufficient to characterize the key node.
- Key node when the underlying probability distribution is uniform.

Disruption (Links Removal)



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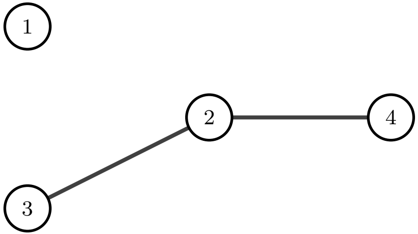

Figure 3: Removal of the links $1 \sim 2$ and $1 \sim 3$

Key Links

- For a realized graph $\tilde{\mathbf{g}}$, let a set of links $L \subseteq E$ be removed and define

$$V(L) := \{i \in N : \exists j \in N \text{ such that } (i,j) \in L\}$$

- The best response of a player $i \in V(L)$ is

$$\tilde{a}_i(\tilde{\mathbf{g}}_i) = 1 + \lambda \sum_{\{j \in N : (i,j) \notin L\}} g_{ij} \sum_{\mathbf{g}_j \in G_j} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i)\, a_j^*(\mathbf{g}_j)$$

- Key Links:

$$\arg\max_{L \subseteq E} \Sigma - \Sigma_{-L}$$

where,

$$\Sigma_{-L} := \sum_{i \in N \setminus V(L)} a_i^*(\tilde{\mathbf{g}}_i) + \sum_{i \in V(L)} \tilde{a}_i(\tilde{\mathbf{g}}_i)$$

Key Links

Theorem 2

If the planner is performing under the constraint that they can remove at most $\bar{l}$ links, then the set of key links L^* are given by

$$L^* \in \arg\max_{\substack{L \subseteq E \\ |L| = \bar{l}}} \lambda \sum_{i \in V(L)} \sum_{j \in L_i} \mathbb{E}_i \left[a_j \mid \tilde{\mathbf{g}}_i \right]$$

where $L_i := \{j \in N : (i,j) \in L\}$

Uniform Case

Lemma 1

When the underlying probability distribution is uniform

(i) For any graph realization with $n \geq 3$ an individual with the highest degree is the key node.

(ii) Any set of $\bar{l}$ links are the key links.

- The expected complementarity that each player expects to extract through their neighbors is the same for everyone.
- The complementarity strength involved in a link is the same for all the links.

Uniform Case

Proposition 1

For any realized graph $\tilde{\mathbf{g}}$ if k^* is the player with the highest degree. Denote by $\Delta\Sigma_{key}$ as the change in aggregate activity in the network due to the removal of k^*. Alternatively, let the disruptor remove all the links of k^*, i.e. $L = \{(k^*, i) : g_{ik^*} = 1\}$, then denote $\Delta\Sigma_{links}$ as the change in aggregate activity due to the removal of the links of k^*. Then we have $\Delta\Sigma_{links} \geq \Delta\Sigma_{key}$.