Should R&D Subsidies be Financed with Public Debt?*

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Abstract

Innovation is a key driver of growth and generates significant positive externalities. This creates a role for government intervention through R&D-enhancing programs such as public R&D subsidies or tax credits. However, while the literature mostly agrees on the positive effects of these programs on innovation and growth, it has not yet given a clear answer on how they should be financed. This paper asks the question whether R&D subsidies should be financed with public debt instead of taxes. The reasoning behind this question is that deficit-financed subsidies improve growth, which makes debt sustainability easier. I first establish that debt financing is not a good idea for economies with a high level of debt as they have higher (r-g), making debt sustainability harder, but may be a viable solution for economies with lower levels of debt. I do so through the lens of a Schumpeterian growth model enhanced with perpetual youth households that allows for dynamic inefficiency. The main mechanism is that high debt crowds out capital, which reduces innovation and future growth. Future work will dive deeper into the question by considering trade-offs with distortionary taxation and a stochastic environment, making it a more appropriate setting to tackle this question using empirical evidence and conducting welfare analysis.

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1 Introduction

Innovation is widely recognized as a key driver of growth. The landmark contributions of Romer (1990); Aghion and Howitt (1992) have greatly enriched our understanding of how innovation propels endogenous growth. They distinguish between horizontal innovation—namely the introduction of entirely new products—and vertical innovation—improvements in the quality or performance of existing goods—and show that both forms are fundamental to economic expansion. Such innovations are typically spurred by R&D investments from firms and entrepreneurs seeking to raise profits or enter new markets.

It is also well established (see Arrow (1972); Aghion and Howitt (1992); Acemoglu (2008); Hall and Lerner (2010); Akcigit et al. (2022a)) that innovation generates positive externalities. A novel idea often finds applications beyond its originator, benefiting other firms, sectors, or the wider economy. Consequently, competitive markets tend to underproduce innovation, creating a justification for government intervention to correct this market failure.

Governments realize this, as well as the importance of innovation for growth. This is why they adopt programs to foster innovation. One way they do so, is by helping startups innovate. In the United States there is the Small Business Innovation Research (SBIR) public program, coordinated by the Small Business Administration, which gives R&D grants to small businesses with the aim of helping them conduct R&D. The SBIR program awards around $4 billion in grants each year to around 4000 small firms[1]. In Europe, there is the EIC Accelerator program, which pays lump sum grants to European Startups under the Horizon Europe program (2021-2027). Some other similar programs are SBRI (UK), Market Development Fund (Denmark) etc.

In addition to public funding for startups, governments use funds for research subsidies and investment in public infrastructure, among other measures, all in the hopes of boosting innovation. This boost comes from the complementarity of public and private resources for R&D. Furthermore, there are R&D tax credit programs that incentivize firms to spend more on R&D. The empirical literature mostly agrees that these programs boost private R&D expenditures (Almus and Czarnitzki, 2003; Rao, 2016; Acconcia and Cantabene, 2018; Dechezleprêtre et al., 2023) and innovation (Jaffe and Le, 2015; Bronzini and Piselli, 2016; Howell, 2017; Dechezleprêtre et al., 2023).

However, while we can say almost with certainty that public R&D subsidies (or R&D tax credits) succeed in incentivizing firms to increase their private R&D expenditures, and thus lead to the development of new and more advanced technologies, the literature still has not provided a clear answer on how to finance these programs. One natural answer is taxation. However, taxation creates distortions. Akcigit and Stantcheva (2020) argue that corporate taxation can lead to a decrease in private R&D expenditure and innovation outputs. One could argue that if we want to boost innovation, households should bear the cost, as taxing firms may lead to the opposite result. But, labor/consumption taxation creates welfare losses. They can also have negative effects on demand which decreases potential profits and thus returns of innovation. Finally, from a political economy perspective, it will generally be easier to pass a program that uses debt financing as it will bear less costs for the voters.

The other option is to use public debt to finance innovation. Mario Draghi, the former prime minister of Italy, during his term (2021-2022), decided to borrow from the EU recovery fund to boost growth through innovation. The plan was to use the funds to make investments in education, research, infrastructure, health, digital sector and decarbonization. The plan did not mention any fiscal consolidation, even though Italy's general government debt to GDP ratio was 154.29% in 2020[2]. At the European level, the European Commission plans defense investments of 800 billion euros with the plan ReArm Europe, and much of it is planned to be raised from the capital markets[3]. The massive amounts of funds needed raised a debate about the tightness of deficits and debt-to-GDP ratios imposed by the Maastricht treaty. Blanchard (2019) argues that historically long-term bond yields have been lower than the growth rates ($r < g$). This essentially implies that the debt-to-GDP ratio can be perpetually rolled over (output increases faster than debt liabilities). Also, R&D and innovation increase the growth rate of the economy. Thus, it is worth exploring if debt financed R&D subsidies can self-finance themselves through the growth they create.

Although the debt financing solution may sound appealing (especially to politicians) it may not be a free lunch. Debt may indeed help avoid the distortions and drops in innovative activity caused by taxes, but it also crowds out physical capital. Howitt and Aghion (1998); Aghion and Howitt (2017) show that more physical capital means more innovation and more growth. Not to mention that at high levels of public debt, fiscal consolidations are more likely, which may discourage private R&D expenditures by firms Akcigit and Stantcheva (2020); Sever (2025). In addition, many studies have shown that debt can significantly inhibit growth (Reinhart and Rogoff, 2010; Woo and Kumar, 2015; Chudik et al., 2017; Sever, 2025).

The purpose of this paper is two-fold. First, it tries to show through the lens of a theoretical model what is the

[1] Source: https://www.sbir.gov/
[2] Source: IMF
[3] https://ec.europa.eu/commission/presscorner/detail/en/ip_25_793

effect of public debt on innovation and growth, and consequently on the potential sustainability of debt-financed R&D subsidies. The novelty of the model, with respect to the previous literature, is that it takes into account crowding-out effects of public debt on physical capital that are a consequence of finite lives. The model is a combination of a perpetual youth model (Blanchard, 1985; Saint-Paul, 1992) and a Schumpeterian growth model with physical capital (Howitt and Aghion, 1998; Aghion and Howitt, 2017). It consists of households who face a fixed probability of dying irrespectively of when they were born. In this model, labor supply is inelastic[4] and agents have a decreasing labor productivity over their lifetime[5]. They also spend a fraction of their labor income on R&D spending through an innovation fund. The fund then distributes the funds to potential innovators who try to improve the quality of the intermediate product and earn monopoly rents. The profits are then distributed back to the households. In this setting, I establish that the higher public debt implies less innovation and increases the difference $r - g$ in the long run, which means that free lunch for subsidies is unlikely. The mechanism behind this result is that high public debt crowds out physical capital, which increases the interest rate but also decreases growth.

Secondly, it aims to answer the question whether R&D subsidies can and should be financed with public debt. The second goal can be divided into two sub-questions. First, is it actually sustainable? That is, if the government decides to increase the deficit to increase innovation will that lead to $r < g$. The main takeaway from the model is that free lunch of subsidies may be impossible for countries with high public debt, as for such economies $r > g$. However, through numerical simulations, I show that low public debt may result in a stable long-term steady state at which $r < g$, so perpetual debt rolling over is feasible. A less demanding answer is to see what happens to the taxation needed for the stabilization of public debt. It is still a very positive thing if debt-financed subsidies imply that lower taxes are needed for debt stabilization. The second sub-question is to ask what are the implications of debt financing R&D subsidies on welfare. In particular, with the addition of endogenous labor, there is a role for debt policy in the Ramsey sense. The Ramsey planner would use debt/taxes to finance subsidies so as to maximize the social welfare of the agents, by taking into account the distortions labor taxation creates.

With all that in mind, future work will include endogenous labor supply and distortionary labor/consumption taxation to study the trade-off between taxes and debt using a utilitarian social welfare function (see Krueger and Ludwig (2016)). I also plan an extension to a stochastic economy where dynamic inefficiency is possible similar to Blanchard (2019). The second extension can separate the interest rate of bonds (risk-free rate) and the risky return of capital. This will be done to give a more definitive answer to whether public programs that aim to boost innovation should be financed by debt or taxes.

Answering these questions is important for many reasons. First, while some theoretical work has been done to show the effects of public debt on growth, very little work has been done to show the effects of debt on growth through the innovation channel. Ferraro and Peretto (2020) study the effects of public debt on innovation and growth using a Schumpeterian growth model. Minami and Horii (2025) they show that under some conditions, public R&D subsidies can be self-financed in a model which is a combination of Blanchard (1985) and Romer (1990), similar to what I am doing. However, none of them consider the crowding-out effects of debt on physical capital and thus the negative effects of debt on growth. Cozzi (2023) is the only work that accounts for capital crowding out effects of public debt in a model with Schumpeterian innovation. He also finds that lowering public debt can increase innovation and growth. However, in his model, debt is used for non-productive government expenditures, ignoring the use of public debt for R&D subsidies. In my model instead, public debt is used to finance only productive R&D expenditure. Also in my model public debt is net wealth which may not need to be repaid back due to finite lives. This increases consumption which in turn decreases capital accumulation. Another distinction is that my model allows for the study of dynamic debt/tax policy.

Secondly, it is important for policy makers to know when debt financing is a viable option to finance growth-promoting innovation plans. As already mentioned, high public debt may crowd out capital enough so that the gap between the interest rate and the growth rate is significantly high. In this scenario, taxes may be a better solution despite the unavoidable welfare losses.

The remainder of the paper is organized as follows. Section 2 discusses related literature. Section 3 presents the theoretical growth model. Section 4 explains some key equilibrium properties and results. Section 5 briefly discusses the planned future extensions. Section 6 concludes.

[4]This is the first planned extension in order to allow for welfare analysis

[5]This is done to allow for the possibility of dynamic inefficiency (see Blanchard (1985); Saint-Paul (1992))

2 Related Literature

This work is related to several strands of literature. First, it is related to the extensive literature that studies the effects of public spending on R&D (subsidies, grants, tax credit policies, etc.). It is well understood that innovation is associated with positive externalities (Arrow (1972); Aghion and Howitt (1992); Hall and Lerner (2010); Akcigit et al. (2022a)), and the literature argues that there is room for government intervention. Therefore, governments should adapt policies that fix the underproduction of innovation.

There is a large body of empirical literature relating public R&D expenditures targeted to the private sector with innovation inputs, namely private R&D expenditure. Almus and Czarnitzki (2003) estimate a positive average causal effect of public R&D subsidies on firm R&D intensity (defined as R&D expenditure over sales). By exploiting some variation in US R&D tax credit program, Rao (2016) studies the effects R&D tax credits have on R&D per sales for a sample between 1981 and 1991. The variation comes from legislative changes which alter the credit rates, corporate tax rates, tax base, qualified R&D expenditures, etc. They find that a 10% increase in tax subsidy increases R&D per sales by 19.8% in the short run. Acconcia and Cantabene (2018) find that a tax credit program on Italian firms in 2009, meant to act as a fiscal stimulus to firms, increased R&D expenditure on firms with high liquidity. However, they also find that there was no such increase in high tech firms' R&D. Similar positive effects on private R&D are found by Dechezleprêtre et al. (2023) for the UK R&D Tax Relief Scheme.

Previous empirical work also shows that public spending on R&D has indeed positive effects on innovation outputs (usually measured in terms of patents, probability of application for a patent, citation weighted patents, etc.). Using data from New Zealand companies, Jaffe and Le (2015) evaluated the effects of public R&D grants on private firm innovation. They show that firms that received the grants had significantly higher probability of applying for a patent and almost double probability of introducing a new product compared to the firms that did not receive the grant. Bronzini and Piselli (2016) show that public R&D grants had strong positive causal effects on small firms' patent applications. Howell (2017) shows that winners of the phase 1 SBIR grant program of the US, experienced positive effects on cite-weighted patents, future revenue. She argues that the reason for that is the grant eases some financial constraints and firms can do product prototyping which convinces venture capitalists to invest on a project thus making patenting easier. Dechezleprêtre et al. (2023) exploit a change in the UK R&D Tax Relief Scheme to causally show that tax incentives have positive long-lasting effects on patents for small-medium enterprises. However, there is also research (Ådne Cappelen, Raknerud and Rybalka, 2012) that shows no significant effect of public spending on R&D on firms' innovation outputs.

This work is also tied to a large body of empirical literature that studies the relationship between debt and growth. Reinhart and Rogoff (2010) show that countries with high public debt (above 90% of GDP) have a lower average growth rate (2% less) than countries with lower debt. This result is robust between emerging and advanced economies. Woo and Kumar (2015), using a panel comprised of 38 advanced and emerging economies, show that a 10% increase in initial debt to GDP ratio results to a 0.2% decrease in the growth rate. They argue that the reason is the crowding out of capital, which results to a decrease in labor productivity. This negative correlation between debt and growth is also found by Chudik et al. (2017). They also find that there is no universal threshold effect under which debt may have positive effects on growth. On the other hand, using panel data on 12 European countries, Checherita-Westphal and Rother (2012) find an inverted U relationship where low levels of public debt are enhancing growth. The model developed below aims to contribute to this literature by providing a theoretical model which replicates to a degree these empirical results.

Next, this work is related to endogenous growth models where the main driver for is R&D and innovation. The model is based on the Schumpeterian Growth theory (Aghion and Howitt, 1992; Howitt and Aghion, 1998; Aghion and Howitt, 2017; Huetsch et al., 2023), rather than the theory of horizontal innovation Romer (1990). In the Schumpeterian growth theory (vertical innovations), economic growth results from quality-improving innovations which replace the existing (lower quality product) earning the innovator monopoly rents. The model developed below is based on Howitt and Aghion (1998); Aghion and Howitt (2017) in order to capture the positive effects of capital accumulation on endogenous growth.

In particular, it is closely related with work that tries to combine innovation and public debt. Teles and Mussolini (2014) show that in a setting of endogenous growth through productive government expenditures, high levels of public debt can impair the effectiveness of public investment on growth. Ferraro and Peretto (2020) use a Schumpeterian growth model to show that high public debt unavoidably leads to fiscal consolidations which decrease growth and permanently lower the output per capita path. Cozzi (2023) discusses the welfare implications of debt, through the lens of an incomplete markets model with Schumpeterian growth, accounting for capital accumulation, calibrated to match US inequality, and finds that eliminating public debt would increase welfare by 0.8% while having a 100% debt to GDP implies a 0.5% welfare loss. He also shows that more public debt decreases growth and increases the interest rate. However, as already mentioned, he does not consider productive government expenditures and the effects of debt on growth work through different channels (less demand for debt

means more demand for other assets which includes capital). The main contribution of this paper at the moment is to provide an analytically tractable dynamic theoretical model that (i) accounts for capital accumulation and (ii) studies the effects of public debt on innovation and growth, which as an immediate implication, affects $r - g$.

Finally, in next stages, this work will be related with three strands of literature. The first one is the public finance literature which uses social welfare functions maximization to determine the optimal use of debt, taxes and subsidies Krueger and Ludwig (2016). The aim is to do that by adding distortionary labor taxes in the model. This will introduce the direct tradeoff between deficit and tax finance.

The second strand of literature is the debt sustainability literature. Blanchard (2019) discusses the possibility of permanent debt roll-overs in the case where the safe interest rate is lower than the growth rate. In a perpetual youth economy with price rigidities, Angeletos et al. (2024), show that deficits can self-finance themselves. They argue, through a theoretical NK model, that the boom generated by the deficit, expands the tax base and increases inflation. Both these factors make debt sustainability easier. Closer to this work, Minami and Horii (2025) show, using a perpetual youth setting with horizontal innovation, that when there are sufficient life-cycle saving motives, permanent debt roll-overs may be possible which essentially makes R&D subsidies a free lunch. A potential future contribution of this project is to contribute to provide a framework that allows the study of the sustainability of public R&D programs through debt financing, while also taking into account capital accumulation.

Lastly, this work will also be tied marginally to the relatively new literature that studies the effects of taxes on innovation. Arnold and Schwellnus (2008) discuss the negative effects of corporate taxation on investments and productivity of firms. Atanassov and Liu (2020) use differences in differences to show that tax cuts have positive effects on patents and patent citations, while tax increases have the opposite effect. Akcigit and Stantcheva (2020); Akcigit et al. (2022b) show that inventor and corporate taxation can have negative effects on innovation. In particular, taxes adversely affect the probability of acquiring a patent and the number of total patents.

3 Model

3.1 Production Sector

For the production side of the economy I consider the typical Schumpeterian environment as in Aghion and Howitt (1992); Howitt and Aghion (1998). There is a unique final good (the numeraire), which is produced by perfectly competitive firms, under a constant returns to scale technology. The inputs for the final good are productive labor L and a continuum of intermediate goods x_i, $i \in [0, 1]$. The intermediate goods market operates under monopolistic competition. Each intermediate input $x_i(t)$ is produced by a local monopolist who uses only physical capital $K_i(t)$ for the production of their intermediate good. Each intermediate input is associated with quality $q_i(t)$. As the quality of an input increases, it is more useful in the production of the final good, thus its demand increases. The engine of growth in this model, is the continuous and endogenously sustained improvement in quality across all sectors.

3.1.1 Final Good Firms

The technology for the production of the final good follows the work of Howitt and Aghion (1998); Aghion and Howitt (2017) and is described by the following production function

$$Y(t) = Z \int_0^1 (q_i(t)L)^{1-\alpha} x_i(t)^{\alpha} di \tag{1}$$

where $q_i(t)$ is the quality of input i, L is labor (which is normalized to 1), $x_i(t)$ is the quantity of input i used in production, and Z is a deterministic scaling parameter.

The final good firms maximize their profits, thus the input demand is given by the input's marginal product, that is

$$w(t) = (1 - \alpha) Z \int_0^1 q_i(t)^{1-\alpha} x_i(t)^{\alpha} \, di \tag{2}$$

$$p_i(t) = \alpha Z q_i(t)^{1-\alpha} x_i(t)^{\alpha-1} \tag{3}$$

Equation (2) gives the wage and equation (3) is the downward slopping demand faced by the intermediate local monopolists.

3.1.2 Intermediate Good Firms

The intermediate good is produced using only one input, physical capital $K_i(t)$. The production function for the intermediate firm i is given by

$$x_i(t) = K_i(t) \tag{4}$$

This implies that the cost function of the intermediate monopolist is $\zeta_i(t)x_i(t)$, where $\zeta(t) = r(t)$ or the price of capital. I have assumed for simplicity that there is no capital depreciation.

The intermediate monopolist i chooses the quantity $x_i(t)$ they produce in order to maximize profits

$$\pi_i(t) = \max_{x_i(t)} \Big[\alpha Z \, q_i(t)^{1-\alpha} \, x_i(t)^{\alpha} \; - \; r(t) \, x_i(t) \Big] \tag{5}$$

The first order condition implies that the optimal quantity of intermediate good $x_i(t)$ is

$$x_i(t) = \left(\frac{\alpha^2}{r(t)} Z \right)^{\frac{1}{1-\alpha}} q_i(t) \tag{6}$$

Define $Q(t) := \int_0^1 q_i(t)di$ to be the aggregate quality (can also be thought as TFP) and $k(t) := K(t)/Q(t)$ to be capital over aggregate quality (can also be thought as capital per efficient labor). Then, following Aghion and Howitt (2017) the interest rate is given by

$$r(t) = \alpha^2 Z k(t)^{\alpha - 1} \tag{7}$$

and the maximized profits for firm i are given by

$$\pi_i(t) = \alpha(1-\alpha) Z q_i(t) k(t)^{\alpha} \tag{8}$$

From the market clearing condition for aggregate capital we have

$$K(t) = \int_0^1 K_i(t)di = \int_0^1 x_i(t)di = \left(\frac{\alpha^2}{r(t)} Z \right)^{\frac{1}{1-\alpha}} \int_0^1 q_i(t)di = \left(\frac{\alpha^2}{r(t)} Z \right)^{\frac{1}{1-\alpha}} Q(t)$$

The last equation combined with (6) imply that the quantity of intermediate input i that is produced is

$$x_i(t) = q_i(t)k(t) \tag{9}$$

Hence, we can see that as an intermediate input becomes more useful in production (or equivalently, it is of higher quality q), its demand increases and therefore more of it is produced.

Given equation (8) we get the following equations for aggregate output Y and wage w respectively

$$Y(t) = ZQ(t)k(t)^{\alpha} \tag{10}$$

$$w(t) = (1-\alpha) ZQ(t)k(t)^{\alpha} \tag{11}$$

The aggregate profits can also be written as

$$\pi(t) := \int_0^1 \pi_i(t)di = \alpha(1-\alpha) ZQ(t)k(t)^{\alpha} \tag{12}$$

An important observation here is that the final good is distributed between labor costs $w(t)L = w(t)$, capital costs $r(t)K_i(t)$, and profits $\pi_i(t)$. Thus, it can be easily verified that

$$Y(t) = \int_0^1 \pi_i(t)di + r(t) \int_0^1 K_i(t)di + w(t)L$$

3.2 Innovation Sector

Innovations improve the existing quality $q_i(t)$ of an input i with a certain probability. If innovation in sector i is successful, it increases the quality of the intermediate product i from $q_i(t)$ to $\lambda q_i(t)$ where $\lambda > 1$ and the successful innovator replaces the previous monopolist. The probability of successful innovation depends positively on R&D expenditures by the innovators. Following Howitt and Aghion (1998); Acemoglu (2008), I assume a probability which is linear on R&D expenditure per current quality of input i, $q_i(t)$, which is given by

$$\phi_i(t|q_i(t)) = \eta \frac{R_i(t)}{q_i(t)} \tag{13}$$

where $R_i(t)$ is the private R&D expenditure by an innovator in sector i.

The innovation sector operates under free entry. It is also assumed that the government subsidizes a portion σ of the R&D expenditure costs of the innovators and that there is an upper bound to how much the subsidy rate $\bar{\sigma} < 1$. The free entry condition in sector i with quality $\lambda^{-1}q_i(t)$ is

$$\phi_i(t|\lambda^{-1}q_i(t))V_i(t|q_i(t)) = (1-\sigma)R_i(t) \Rightarrow \frac{\eta}{\lambda^{-1}q_i(t)}V_i(t|q_i(t)) = 1-\sigma \tag{14}$$

Appendix A.2 shows that the value of a firm i is given by

$$V_i(t|q_i(t)) = \frac{\pi_i(t)}{r(t)+\phi_i(t|q_i(t))} = \frac{\alpha(1-\alpha)Zq_i(t)k(t)^\alpha}{r(t)+\phi_i(t|q_i(t))} \tag{15}$$

Combining equations (14) and (15) we have that the probability of successful innovation is

$$\phi(k(t);\sigma) := \phi_i(t|q_i(t)) = \Big[\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t) - \alpha^2\Big]Zk(t)^{\alpha-1} \tag{16}$$

As in Howitt and Aghion (1998); Aghion and Howitt (2017), in partial equilibrium the probability of successful innovation is the same for all sectors and it positively depends on the government subsidy rate σ, the step of innovation λ. Given (16) we can calculate $R_i(t)$ using (13). Thus, the $R_i(t)$ expenditure of innovators in sector i with quality $q_i(t)$ is

$$R_i(t) = \frac{1}{\eta}\phi_i(t|q_i(t))q_i(t) \tag{17}$$

and aggregate R&D expenditure is

$$R(t) = \int_0^1 R_i(t)di = \frac{1}{\eta}\phi(k(t);\sigma)Q(t) \tag{18}$$

Equation (17) implies that, similar to the probability of successful innovation ϕ, the private R&D expenditure depends positively on the subsidy rate σ and the innovation step λ, and it depends negatively on the cost of borrowing capital r.

Appendix A.2. also shows that the growth rate of aggregate productivity Q (and hence output Y along the Balanced Growth Path) is given by

$$g(k(t);\sigma) := \frac{\dot{Q}(t)}{Q(t)} = (\lambda-1)\phi(k(t);\sigma) = (\lambda-1)\Big[\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t) - \alpha^2\Big]Zk(t)^{\alpha-1} \tag{19}$$

3.3 Households

For the households, I consider a perpetual youth environment in the spirit of Blanchard (1985)[6]. Each household is comprised of 1 agent. Agents face a constant exogenous probability of death μ, therefore the average lifetime $(1/\mu)$[7]. The population is $N = \int_{-\infty}^{t} \mu e^{-\mu(t-s)}ds = 1$. The lifetime utility of a household born in time s from period t onwards is

$$E_t\left[\int_t^\infty log(c(s,\tau))e^{-\rho(\tau-t)}d\tau\right], \quad \rho \geq 0$$

[6] This is done in order to break the Ricardian Equivalence

[7] Average time alive is $\int_0^\infty t\mu e^{-\mu t}dt = 1/\mu$

Taking into account the constant probability of death, and the fact that the only source of uncertainty is the time of death the objective function of an s cohort household at time t is given by

$$\int_t^\infty log(c(s,\tau))e^{-(\mu+\rho)(\tau-t)}dv \tag{20}$$

The budget constraint of the household born at time s is given by

$$\dot{a}(s,t) = (r(s,t)+\mu)a(s,t) + (1-\tau(t))\xi e^{-\beta(t-s)}w(t) + \xi e^{-\beta(t-s)}\Big[\pi(t) - (1-\sigma)R(t)\Big] \tag{21}$$

Agents supply labor inelastically. They are also assumed to experience a decrease in labor productivity. In particular, each household starts its life with some productivity $\xi > 1$ which declines at a rate β with age. This, as in Blanchard (1985), implies that in terms of wage a household born at time s earns the labor income $\xi e^{-\beta(t-s)}w(t)$. While this assumption is unrealistic, on the aggregate, it can thought as capturing "saving for retirement motives". It should be noted that this assumption is only made to allow dynamic inefficiency ($r < g$).

For simplicity, I keep the innovation sector exogenous to the households. Following Nuno (2011), it is assumed that in this economy there exists an "innovation fund" which collects resources from households in units of the final good. These resources are then distributed to innovators in each sector i who in turn use the resources for R&D in order to become the next intermediate monopolists[8]. Then, the fund transfers the monopoly profits back to the households in the form of lump sum transfers[9]. It is also assumed that households participate by supplying and receiving transfers from the innovation fund proportional to their labor productivity[10].

Following Blanchard (1985); Saint-Paul (1992); Minami and Horii (2025), I normalize the aggregate productivity of labor to 1. That is, we want

$$\int_{-\infty}^t \xi e^{-\beta(t-s)}\mu e^{-\mu(t-s)}ds = 1 \Rightarrow \xi = \frac{\beta+\mu}{\mu}$$

Define $f(t) := (1-\tau(t))w(t) + [\pi(t) - (1-\sigma)R(t)]$. Then the budget constraint of cohort s ad time t can be compactly written as

$$\dot{a}(s,t) = (r(s,t)+\mu)a(s,t) + \xi e^{-\beta(t-s)}f(t) \tag{22}$$

We also assume a transversality condition.

$$lim_{T\to\infty} e^{-\int_t^T (r(v)+\mu)dv}a(s,T) = 0$$

The solution to the household maximization problem yields the following Euler Equation

$$\frac{\dot{c}(s,t)}{c(s,t)} = r(t) - \rho \tag{23}$$

which in turn implies the following consumption function

$$c(s,t) = (\rho+\mu)\left(a(s,t) + h(s,t)\right) \tag{24}$$

where $h(s,t) := \xi e^{-\beta(t-s)}\int_t^\infty f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}d\tau$ is the human wealth at time t of a household born at time s. The aggregate variables can be computed by integrating over surviving cohorts up until time t. That is

$$X(t) = \int_{-\infty}^t x(s,t)\mu e^{-\mu(t-s)}ds$$

Therefore, by aggregating the consumption function we have

$$C(t) = (\rho+\mu)\left(A(t) + H(t)\right)$$

similarly to Blanchard (1985), the aggregate human wealth is defined as

$$H(t) = \int_{-\infty}^t \mu e^{-\mu(t-s)}\xi e^{-\beta(t-s)}\int_t^\infty f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}d\tau ds = \int_t^\infty f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}d\tau$$

[8] Implicitly it is assumed that the innovators ask the specific R_i such that (14) is satisfied, then the fund extracts this amount from the households

[9] Equivalently, following Huetsch et al. (2023), I could assume that the innovators consume the difference between profits and R&D expenditures, arriving at the same aggregate market structure.

[10] This is done for algebraic simplicity

Differentiating with respect to t and using Leibniz's rule yields

$$\dot{H}(t) = [r(t) + \mu + \beta]H(t) - f(t) \tag{25}$$

The aggregate wealth is given by

$$A(t) = \int_{-\infty}^{t} a(s,t)\mu e^{-\mu(t-s)} ds$$

Taking the derivative with respect to t and using the fact that $a(t,t) = 0$ yields

$$\dot{A}(t) = a(t,t) - \mu A(t) + \int_{-\infty}^{t} \dot{a}(s,t)\mu e^{-\mu(t-s)} ds \Rightarrow$$

$$\dot{A}(t) = r(t)A(t) + f(t) - C(t)$$

The system of dynamic equations is

$$C(t) = (\rho + \mu)(A(t) + H(t)) \tag{26}$$

$$\dot{H}(t) = (r(t) + \mu + \beta)H(t) - f(t) \tag{27}$$

$$\dot{A}(t) = r(t)A(t) + f(t) - C(t) \tag{28}$$

Combining them yields the system

$$\dot{C}(t) = (r(t) + \beta - \rho)C(t) - (\mu + \beta)(\mu + \rho)A(t) \tag{29}$$

$$\dot{A}(t) = r(t)A(t) + f(t) - C(t) \tag{30}$$

3.4 Government

The government finances productive public expenditure on R&D, $G(t) = \sigma R(t)$. It does so by issuing debt $B(t)$ in the form of one-period bonds, and by collecting taxes on labor income from the households. The taxes are different because of the idiosyncratic productivity of the different cohorts, but they aggregate to $T(t) = \tau(t)w(t)$. By the usual no-arbitrage argument, the interest rate on government bonds is the same as the return of capital. The government budget constraint is given by

$$\dot{B}(t) = r(t)B(t) + \sigma R(t) - \tau(t)w(t) \tag{31}$$

3.5 Competitive Equilibrium

The households save on physical capital and government debt. Thus, the market clearing condition is given by

$$A(t) = K(t) + B(t) \tag{32}$$

Substituting into equation (30) yields

$$\dot{K}(t) + \dot{B}(t) = r(t)\Big[K(t) + B(t)\Big] + (1 - \tau(t))w(t) + \pi(t) - (1 - \sigma)R(t) - C(t) \tag{33}$$

then by using the budget constraint of the government yields and using the fact that profits plus labor and capital earnings equals output we have

$$\dot{K}(t) = Y(t) - R(t) - C(t) \tag{34}$$

After substituting the market clearing into the Euler equation (29) it becomes

$$\dot{C}(t) = (r(t) + \beta - \rho)C(t) - (\mu + \beta)(\mu + \rho)[K(t) + B(t)] \tag{35}$$

Next, we calculate the dynamics of the economy in terms of capital per aggregate quality $k(t) := K(t)/Q(t)$ and consumption per aggregate quality $c(t) := C(t)/Q(t)$[11]. The resulting equations are

[11] for $x := X/Q$, $\dot{x} = \dot{X}/Q - x\dot{Q}/Q$

$$\dot{k}(t) = y(t) - \frac{R(t)}{Q(t)} - c(t) - g(k(t);\sigma)k(t) \tag{36}$$

$$\dot{c}(t) = (r(t) + \beta - \rho - g(k(t);\sigma))c(t) - (\mu+\beta)(\mu+\rho)[k(t)+b(t)] \tag{37}$$

Finally, we have the dynamic equation for the evolution of the government budget constraint.

$$\dot{b}(t) = [r(t) - g(k(t);\sigma)]b(t) + \sigma\frac{R(t)}{Q(t)} - \tau(t)\frac{w(t)}{Q(t)} \tag{38}$$

Equations (36) and (37) describe the dynamics of this economy in general equilibrium.

4 Steady State Properties under Fixed debt

Now I look at the equilibrium effects of debt in this economy. Substituting (7),(10),(19) and (18) into (36) and (37) yields the dynamic system only in terms of k, c, b and the fundamentals.

$$\dot{k}(t) = -Z(\lambda-1)\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t)^{\alpha+1} + Z\Big(1 - \frac{\alpha(1-\alpha)\lambda}{1-\sigma} + (\lambda-1)\alpha^2\Big)k(t)^{\alpha} + \frac{Z\alpha^2}{\eta}k(t)^{\alpha-1} - c(t) \tag{39}$$

$$\dot{c}(t) = \Big[-Z(\lambda-1)\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t)^{\alpha} + Z\alpha^2\lambda k(t)^{\alpha-1} + \beta - \rho\Big]c(t) - (\mu+\beta)(\mu+\rho)\big[k(t)+b(t)\big] \tag{40}$$

Assume for now that the government perfectly stabilizes the debt level per aggregate quality at b^*. It does so by changing the tax rates $\tau(t)$ accordingly so that $\dot{b}(t) = 0$, $\forall t$. Then equation (38) can be dropped. The following proposition shows that this model has at least one steady state.

Proposition 1. *Let the economy be characterized by equations (39) and (40). Assume that* $k(0) \geq \underline{k} := \frac{\alpha(1-\sigma)}{\eta\lambda(1-\alpha)}$ *and the stabilized government debt is* $b^* < \Big[\frac{\alpha^2 Z^2}{(\mu+\beta)(\mu+\rho)\underline{k}^{3-2\alpha}} - 1\Big] := \bar{b}$. *Then,* $\exists\bar{k} \in (\underline{k}, +\infty)$ *such that the economy has* ***at least one steady state*** $(k^*, c^*) >> 0$ *with* $k* \in (\underline{k}, \bar{k})$.

Proof. Appendix A.5 □

Unfortunately, the degree of nonlinearity of equation (38) makes it impossible to guarantee uniqueness. However, for all parameter trials, the model seems to have only one positive steady state. Although this does not formally prove uniqueness it shows that the model behaves well, and makes consistent predictions about the equilibrium.

Regarding saddle path stability, it is not possible to show analytically that a steady state exhibits the property. Whether the Blanchard-Khan conditions hold, depends on the parameter values for $\beta, \eta, \lambda, \sigma, Z$. It should also be taken into account that for normal parameter values the model exhibits saddle-path stability. An equilibrium with saddle-path stability is displayed in Figure 1(a).

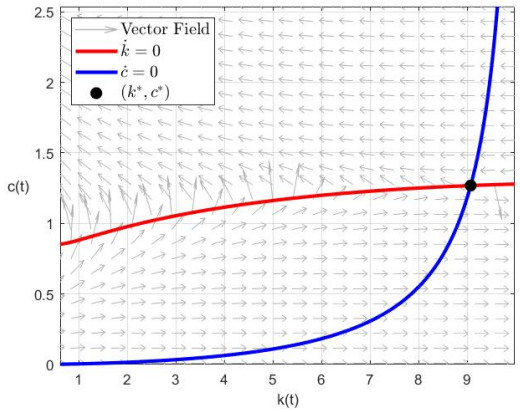
We can still derive some properties for all saddle-path stable steady states. The following proposition describes how an increase at the steady state level of debt affects steady state capital level, and hence long run $r - g$, if that steady state is characterized by saddle path stability.

Proposition 2. *Let* (k^*, c^*) *be a steady state characterized by saddle-path stability. Then an increase in the stabilized level of debt per aggregate quality* b^* *decreases the steady state level of capital* k^*. *This also decreases* $\phi^*(k^*;\sigma)$ *and increases the difference* $r^* - g^*$.

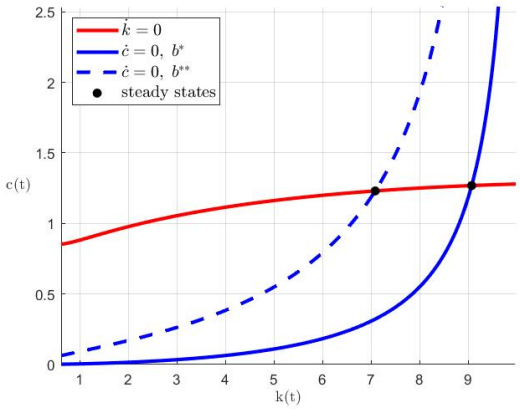
Proof. Appendix A.6 □

Proposition 2 states that an economy with high debt may not be able to rollover debt, if they decide to drop taxes to finance R&D. That experiment assumes that governments, starting from the steady state, increase b^* at time t_0 and then pay the interest on debt by adjusting taxes $\tau(t)$.

The comparative statics behind this result are displayed in Figure 1(b). Intuition is straightforward. An economy with high debt spends less resources on capital accumulation (the capital crowding-out effect) which increases the real interest rate. This also reduces future monopoly profit streams and hence incentives for private R&D



(a) Phase diagram and Directional arrows



(b) Effects of an increase of b^* from b^* to b^{**}

Figure 1: Dynamics of the economy. Figure 1(a) displays an economy with saddle path stability. Even with the degree of non linearity uniqueness is the norm. Figure 1(b) shows the comparative statics effects of increasing government debt. The $\dot{c} = 0$ locus shifts to the left leading to lower k^* and thus lower ϕ^* and higher $r^* - g^*$.

by the potential innovators, which in turn decreases the growth rate. Hence, for an economy with debt, long-run $r - g$ is higher.

The main takeaway from this model is that the amount of current public debt may make debt financing of innovation not sustainable. At some point, it is quite likely that fiscal consolidations will have to be made. Table 1 shows the long-run $r - g$ for different values of B^*/Y^*. It can bee seen that higher levels of debt imply less innovation (smaller ϕ). It can also be seen that lower values of B^*/Y^* can lead the long run interest rate to be lower than the growth rate $r < g$. So, in that case, debt-rolling over is possible. The parameters for the experiment are displayed in Table 2. They have been chosen so that key aggregate such as capital to output ratio, consumption output ratio and growth rate.

B^*/Y^*	k^*	C^*/Y^*	r^*	ϕ^*	g^*	$r^* - g^*$	K^*/Y^*	Stability
0%	2.1646	0.5631	0.0332	**0.2611**	0.0444	−0.0112	3.3467	Yes
25%	2.0327	0.5757	0.0346	**0.2536**	0.0431	−0.0085	3.2093	Yes
50%	1.9085	0.5879	0.0361	**0.2461**	0.0418	−0.0057	3.0772	Yes
100%	1.6715	0.6121	0.0394	**0.2305**	0.0392	0.0003	2.8169	Yes
150%	1.4588	0.6355	0.0432	**0.2148**	0.0365	0.0067	2.5725	Yes
200%	1.2674	0.6585	0.0474	**0.1988**	0.0338	0.0136	2.3423	Yes

Table 1: Steady-state results for various debt-to-quality ratios B^*/Y^*. It can be seen that increasing public debt, decreases the flow rate of innovation and growth while it increases the interest rate. This leads $r - g$ to be higher for higher levels of debt.

Parameter	Description	Value	Source
α	Capital share	1/3	Standard
λ	Innovation step size	1.17	Ad hoc assumption
η	R&D productivity per quality	1.40	Ad hoc assumption
σ	Subsidy rate	0.20	Ad hoc assumption
ρ	Discount rate (time preference)	0.01	Standard (implied discount factor 0.99)
μ	Poisson death rate	0.10	Blanchard (1985)
β	Labor productivity depreciation rate	0.25	Ad hoc assumption
Z	Output scaling factor	0.50	Ad hoc assumption

Table 2: Model parameter values and their sources

5 Future Extensions

Although the model shows that economies with higher debt may have $r > g$, which in turn implies that issuing extra debt for R&D subsidies may not be sustainable, it still does not answer the question of whether tax financing is better in that scenario. It may well be the case that taxation creates large distortions to be optimal. In order to get a more convincing answer the model needs to be enhanced in two fronts.

The first planned extension is the addition of endogenous labor supply. In the modeling framework presented in Section 3, there are two realistic ways the government can raise tax revenues. The first one is to tax the intermediate monopolists. However, this decreases incentives for R&D as argued by Akcigit and Stantcheva

(2020), and since all sectors have the same innovation step we cannot levy taxes on the less productive and subsidize the more productive as argued in Akcigit et al. (2022a). Hence, the only remaining option is to tax labor income. However, in order to do so, we must have endogenous labor choice to properly account for distortions. Then the plan is to use the combination of debt and taxes for the financing of subsidies in order to maximize social welfare. This will be done by using a utilitarian social welfare function in the spirit of Krueger and Ludwig (2016).

An interesting experiment can be performed in that setup. We can study the effects of deficit-financed subsidies on the labor taxes that are needed to stabilize public debt. If more deficit-financed subsidies decrease taxation, even without fully self-financed subsidies, there are important policy implications. However, due to the resource constraint, even without the breaking of the Ricardian Equivalence, an increase in R&D unavoidably reduces the resources that go to capital accumulation and therefore, it may reduce growth. So even in that case there are obvious trade-offs.

The second extension follows the work of Blanchard (2019). Blanchard studies the possibility of debt-rollovers using a two-period overlapping generations model in a stochastic environment with risky capital and risk-less bonds and no growth. He finds that debt rolling over is possible when the risky interest rate is low, since that implies the risk-free rate is also low. In his work, the initial level of debt matters as it determines the risky interest rate path. Higher levels of initial debt makes rolling over more difficult.

However, he abstracts from productive government spending and fiscal consolidations. A natural next step is to enhance his model with productive government expenditure which will be public R&D subsidies. The R&D subsidies will again increase the growth rate. In the case where the government can no longer roll-over debt tax increases will have to be used so that debt does not explode. This enhanced framework will allow for more realistic calibrations which will allow to match aggregate data.

In terms of matching the theory to the data, the best experiment would be to estimate effects of debt financed R&D subsidies aimed at firms (startups). The effects can be entry, patents (or similar measures). Then the model could be enhanced[12] so that the flow rate much its data counterpart. Alternatively, we could identify effects of such programs directly on growth and try to much that with the data.

The ideal way to conduct the experiment is to use text analysis on government announcements that describe increases in public R&D spending for innovation but mention no fiscal consolidation, or match changes to the programs to changes in the government balance.

6 Conclusion

This project aims to answer the question on whether public R&D subsidies should be financed with public debt instead of taxes. The motivation behind this question is based on many dimensions. First, the negative effects of corporate taxes on incentives for private R&D which slow down growth by hindering innovation. Second, there are the negative effects of distortionary labor/consumption taxes on welfare. Lastly, as argued, one way to avoid these growth-reducing distortions could be the use of public debt to finance these public R&D subsidy programs.

To that end, I developed a theoretical model, which for the moment, abstracts from endogenous labor and labor income taxation and focuses only on the effects of public debt on innovation and long-run growth. The model shows that in all stable equilibria, the higher the level of public debt is, the greater the difference $r - g$ is (higher interest rate, lower growth rate). The reasoning behind this stems from the crowding-out effects debt has on physical capital. More importantly, because the model allows for dynamic inefficiency, an economy with a low enough debt may have the long-run growth rate be higher than the long-run interest rate. In this case, the government can perpetually roll-over debt and there is no need for taxation thus creating the possibility of self-financed subsidies.

To make the results more realistic and relevant to policy making, two theoretical extensions are planned. The first is to transform the above model by adding a stochastic environment while still allowing for dynamic inefficiency. The other extension is adding proportional labor taxation and conducting welfare analysis by comparing debt and tax financing of the R&D subsidies.

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Appendix

A.1 - Firm Optimality and Aggregation

Intermediate firm i solves

$$\pi_i(t) = max_{x_i(t)} \alpha Z q_i(t)^{1-\alpha} x_i(t)^{\alpha} - r(t)x_i(t) \tag{41}$$

The first order condition is

$$\alpha^2 Z q_i(t)^{1-\alpha} x_i(t)^{\alpha-1} = r(t) \Rightarrow$$

$$x_i(t) = \left(\frac{\alpha^2}{r(t)} Z\right)^{\frac{1}{1-\alpha}} q_i(t) \tag{42}$$

Next, as noted in the text,

$$K(t) = \left(\frac{\alpha^2}{r(t)} Z\right)^{\frac{1}{1-\alpha}} Q(t) \Leftrightarrow$$

$$q_i(t)\frac{K(t)}{Q(t)} = \left(\frac{\alpha^2}{r(t)} Z\right)^{\frac{1}{1-\alpha}} q_i(t)$$

then, using (6) we have

$$x_i(t) = q_i(t)k(t) \tag{43}$$

Substituting this into the FOC we find the interest rate

$$r(t) = \alpha^2 Z q_i(t)^{1-\alpha} x_i(t)^{\alpha-1} \Rightarrow$$

$$r(t) = \alpha^2 Z q_i(t)^{1-\alpha} q_i(t)^{\alpha-1} k(t)^{\alpha-1} \Rightarrow$$

$$r(t) = \alpha^2 Z k(t)^{\alpha-1} \tag{44}$$

Substituting this solution to the profits yields

$$\pi_i(t) = \alpha Z q_i(t)^{1-\alpha} q_i(t)^{\alpha} k(t)^{\alpha} - \alpha^2 Z k(t)^{\alpha-1} q_i(t) k(t) \Rightarrow$$

$$\pi_i(t) = \alpha(1-\alpha) Z q_i(t) k(t)^{\alpha} \tag{45}$$

Thus, the aggregate profits are

$$\pi(t) = \int_0^1 \pi_i(t) di = \alpha(1-\alpha) Z Q(t) k(t)^{\alpha}$$

Next, note that

$$\pi(t) + w(t)L + r(t)K(t) = \alpha(1-\alpha) Z Q(t) \Big(\frac{K(t)}{Q(t)}\Big)^{\alpha} + (1-\alpha) Z Q(t) \Big(\frac{K(t)}{Q(t)}\Big)^{\alpha} + \alpha^2 Z \Big(\frac{K(t)}{Q(t)}\Big)^{\alpha-1} K(t) =$$

$$(\alpha - \alpha^2) Z Q(t) k(t)^{\alpha} + (1-\alpha) Z Q(t) k(t)^{\alpha} + \alpha^2 Z Q(t) k(t)^{\alpha} = Z Q(t) k(t)^{\alpha} = Y(t)$$

A.2 - Innovation Sector

The value of a firm i is given by the discounted value of future profits. The discount rate is the interest rate $r(t)$ plus the probability of being replaced by another innovator in the future $\phi_i(t)$. Thus, the value of firm i can be written as

$$V_i(t|q_i(t)) = \int_t^\infty e^{-\int_t^\tau r(\nu)+\phi_i(\nu)d\tau}\pi_i(\tau)d\tau = \int_t^\infty e^{-\int_t^\tau r(\nu)+\phi_i(\nu)d\tau}\alpha(1-\alpha)Zq_i(t)k(\tau)^\alpha d\tau$$

Define $v_i(t) := V_i(t)/q_i(t)$ the value over quality for firm i. Note that $q_i(t)$ will remain constant up until a new innovator successfully innovates, in which case the current firm is replaced. Therefore, we can divide both sides with $q_i(t)$ to get

$$v_i(t|q_i(t)) = \int_t^\infty e^{-\int_t^\tau r(\nu)+\phi_i(\nu)d\tau}\alpha(1-\alpha)Zk(\tau)^\alpha d\tau$$

Differentiating with respect to t and using Leibniz's rule yields the Hamilton-Jacobi-Bellman equation

$$\dot{v}_i(t|q_i(t)) = 0 - \alpha(1-\alpha)Zk(t)^\alpha + \int_t^\infty \frac{d}{dt}\Big[e^{-\int_t^\tau r(\nu)+\phi_i(\nu)d\tau}\alpha(1-\alpha)Zk(\tau)^\alpha\Big]d\tau \Rightarrow$$

$$\dot{v}_i(t|q_i(t)) = 0 - \alpha(1-\alpha)Zk(t)^\alpha + \int_t^\infty [r(t)+\phi_i(t)]e^{-\int_t^\tau r(\nu)+\phi_i(\nu)d\tau}\alpha(1-\alpha)Zk(\tau)^\alpha d\tau \Rightarrow$$

$$\dot{v}_i(t|q_i(t)) = [r(t)+\phi_i(t)]v_i(t|q_i(t)) - \alpha(1-\alpha)Zk(t)^\alpha$$

The free entry condition can be written as

$$\eta\frac{R_i(t)}{\lambda^{-1}q_i(t)}V_i(t|q_i(t)) = \eta\lambda R_i(t)v_i(t) = (1-\sigma)R_i(t) \Rightarrow v_i(t) = \frac{1-\sigma}{\eta\lambda}$$

so v_i is a constant and therefore $\dot{v}_i = 0$. Using that along with the HJB yields

$$V_i(t|q_i(t)) = \frac{\pi_i(t)}{r(t)+\phi_i(t)}$$

Next, I calculate the aggregate growth rate for the aggregate productivity. Given that the probability of successful innovation is $\phi(k(t);\sigma)$ we have that at time Δt a fraction $\phi(k(t);\sigma)\Delta t$ of innovators succeed, and the rest $(1-\phi(k(t);\sigma)\Delta t)$ fail. This implies that a fraction $\phi(k(t);\sigma)\Delta t$ of the aggregate quality is upgraded to $\lambda Q(t)$ and the rest remains at $Q(t)$. Therefore for an interval Δt the law of motion of aggregate quality $Q(t)$ is

$$Q(t+\Delta t) = \lambda\phi(k(t);\sigma)\Delta t Q(t) + \Big[1-\phi(k(t);\sigma)\Delta t\Big]Q(t) \Rightarrow$$
$$\frac{Q(t+\Delta t)-Q(t)}{\Delta t} = \frac{(\lambda-1)\phi(k(t);\sigma)\Delta t}{\Delta t}Q(t) \Rightarrow$$

As $\Delta t \to 0$, the growth rate of aggregate quality Q is

$$lim_{\Delta t\to 0}\frac{Q(t+\Delta t)-Q(t)}{\Delta t} = (\lambda-1)\phi(k(t);\sigma)Q(t) \Rightarrow$$
$$g(k(t);\sigma) := \frac{\dot{Q}(t)}{Q(t)} = (\lambda-1)\phi(k(t);\sigma) \tag{46}$$

In the Balanced Growth Path, the growth rate of Q and Y coincide and it is

$$g^* := g(k^*(\sigma);\sigma) = (\lambda-1)\phi(k^*(\sigma);\sigma)$$

A.3 Household Optimization

The current value Hamiltonian is given by

$$H = log(c(s,t)) + \theta(s,t)[(r(t)+\mu)a(s,t) + \xi e^{-\beta(t-s)}f(t) - c(s,t)]$$

The first order conditions are

$$\frac{\partial H}{\partial c(s,t)} = 0 \Rightarrow \frac{1}{c(s,t)} = \theta(s,t) \Rightarrow \dot{\theta}(s,t) = -\frac{\dot{c}(s,t)}{c(s,t)^2}$$

$$\frac{\partial H}{\partial a(s,t)} = (\rho+\mu)\theta(s,t) - \dot{\theta}(s,t) \Rightarrow (r(t)+\mu)\theta(s,t) = (\rho+\mu)\theta(s,t) - \dot{\theta}(s,t) \Rightarrow$$

$$(r(t)-\rho)\theta(s,t) = -\dot{\theta}(s,t)$$

Combining the FOCs yields the Euler equation

$$\dot{c}(s,t) = (r(t)-\rho)c(s,t)$$

The Euler equation can be solved to give consumption at time τ given consumption at time t.

$$c(s,\tau) = c(s,t)e^{\int_t^\tau (r(v)-\rho)dv}$$

Next, we turn to the individual budget constraint (22). Define the integration function $I(t)$ such that $\dot{I}(t) = -(r(t)+\mu)I(t)$. Multiply both sides of the budget constraint at time τ with $I(t)$ and rearrange to get

$$\dot{a}(s,t)I(t) - (r(t)+\mu)I(t)a(s,t) = \xi e^{-\beta(t-s)}f(t)I(t) - c(s,t)I(t) \Rightarrow$$

$$\dot{a}(s,t)I(t) - \dot{I}(t)a(s,t) = \xi e^{-\beta(t-s)}f(t)I(t) - c(s,t)I(t) \Rightarrow$$

$$\frac{d}{dt}\Big[a(s,t)I(t)\Big] = \xi e^{-\beta(t-s)}f(t)I(t) - c(s,t)I(t) \Rightarrow$$

$$\int \frac{d}{dt}\Big[a(s,t)I(t)\Big]dt = \int \xi e^{-\beta(t-s)}f(t)I(t)dt - \int c(s,t)I(t)dt \Rightarrow$$

$$\int_t^T \frac{d}{d\tau}\Big[a(s,\tau)I(\tau)\Big]dt = \int_t^T \xi e^{-\beta(\tau-s)}f(\tau)I(\tau)dt - \int_t^T c(s,\tau)I(\tau)dt \Rightarrow$$

Also, note that the definition of $I(t)$ we have $I(\tau) = e^{-\int_t^\tau (r(v)+\rho)dv}$. So, the budget constraint becomes

$$a(s,T)e^{-\int_t^\tau (r(v)+\rho)dv} - a(s,t) = \int_t^T \xi e^{-\beta(\tau-s)}f(\tau)e^{-\int_t^\tau (r(v)+\rho)dv}dt - \int_t^T c(s,\tau)e^{-\int_t^\tau (r(v)+\rho)dv}dt \Rightarrow$$

$$a(s,T)e^{-\int_t^\tau (r(v)+\rho)dv} - a(s,t) = \xi e^{-\beta(t-s)}\int_t^T f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}dt - \int_t^T c(\tau)e^{-\int_t^\tau (r(v)+\rho)dv}dt \Rightarrow$$

Taking the limit as $T \to \infty$ and using the household's transversality condition for terminal wealth we have

$$\int_t^T c(\tau)e^{-\int_t^\tau (r(v)+\rho)dv}dt = \xi e^{-\beta(t-s)}\int_t^T f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}dt + a(s,t)$$

Now, substitute the solution to the consumption $c(s,\tau)$ to the lifetime budget constraint.

$$c(s,t)\int_t^T e^{\int_t^\tau (r(v)-\rho)dv}e^{-\int_t^\tau (r(v)+\rho)dv}dt = \xi e^{-\beta(t-s)}\int_t^T f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}dt + a(s,t) \Rightarrow$$

$$c(s,t)\int_t^T e^{-(\mu+\rho)(\tau-t)}dt = \xi e^{-\beta(t-s)}\int_t^T f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}dt + a(s,t) \Rightarrow$$

$$c(s,t) = (\rho+\mu)\Big(\xi e^{-\beta(t-s)}\int_t^T f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}d\tau + a(s,t)\Big)$$

or it can be written as in (24) using $h(s,t) := \xi e^{-\beta(t-s)}\int_t^T f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}d\tau$. Next, we aggregate the consumption function as $T \to \infty$

$$C(t) = \int_{-\infty}^t c(s,t)\mu e^{-\mu(t-s)}ds = \int_{-\infty}^t (\rho+\mu)\Big(\xi e^{-\beta(t-s)}\int_t^\infty f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}d\tau + a(s,t)\Big)\mu e^{-\mu(t-s)}ds =$$

$$(\rho+\theta)\int_{-\infty}^t \mu e^{-\mu(t-s)}a(s,t)ds + (\rho+\theta)\int_{-\infty}^t \xi e^{-\beta(t-s)}\Big(\int_t^\infty f(\tau)e^{-\int_t^\tau (r(v)+\rho+\beta)dv}d\tau\Big)\mu e^{-\mu(t-s)}ds = (\rho+\theta)(A(t)+H(t))$$

A.4 Derivation of Dynamic Equations for the Model

For equation (39) which is the equation for the law of motion of capital use equations (10),(16),(17) and (19) to get

$$\dot{k}(t) = Zk(t)^\alpha - \frac{1}{\eta}Z\Big[\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t)^\alpha - \alpha^2 k(t)^{\alpha-1}\Big] - (\lambda-1)Z\Big[\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t)^\alpha - \alpha^2 k(t)^{\alpha-1}\Big]k(t) - c(t) \Rightarrow$$

$$\dot{k}(t) = -(\lambda-1)Z\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t)^{\alpha+1} + \Big[1 - \frac{\alpha(1-\alpha)\lambda}{1-\sigma} + (\lambda-1)\alpha^2\Big]k(t) + \frac{Z\alpha^2}{\eta}k(t)^{\alpha-1} - c(t)$$

Then, for equation (40), use equations (7) and (19) to get

$$\dot{c}(t) = \Big(\alpha^2 Zk(t)^{\alpha-1} + \beta - \rho - (\lambda-1)Z\Big[\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t)^\alpha - \alpha^2 k(t)^{\alpha-1}\Big]\Big)c(t) + (\mu+\beta)(\mu+\rho)[k(t)+b(t)] \Rightarrow$$

$$\dot{c}(t) = \Big[-(\lambda-1)Z\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t)^\alpha + \lambda\alpha^2 Zk(t)^{\alpha-1} + \beta - \rho\Big]c(t) + (\mu+\beta)(\mu+\rho)[k(t)+b(t)]$$

A.5 Proof of Proposition 1

The steady state is characterized by $\dot{c}(t) = \dot{k}(t) = 0$. Imposing this on the system of equations (38) and (39) yields

$$c \;=\; Z\,k^\alpha \;-\; Z\,\frac{1}{\eta}\Big[\frac{\alpha(1-\alpha)\,\eta\,\lambda}{1-\sigma}\,k^\alpha \;-\; \alpha^2\,k^{\alpha-1}\Big] \;-\; Z\,(\lambda-1)\Big[\frac{\alpha(1-\alpha)\,\eta\,\lambda}{1-\sigma}\,k^{\alpha+1} \;-\; \alpha^2\,k^\alpha\Big] \Rightarrow$$

$$c_k(k) = -Z(\lambda-1)\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k^{\alpha+1} + Z\Big(1 - \frac{\alpha(1-\alpha)\lambda}{1-\sigma} + (\lambda-1)\alpha^2\Big)k^\alpha + Z\frac{\alpha^2}{\eta}k^{\alpha-1}$$

where $c_k(k)$ is the function that describes the nullcline for $\dot{k} = 0$ and

$$c \;=\; \frac{(\mu+\beta)\,(\mu+\rho)\,\big[k+b(t)\big]}{\alpha^2\,Z\,k^{\alpha-1} \;-\; Z\,(\lambda-1)\Big[\dfrac{\alpha(1-\alpha)\,\eta\,\lambda}{1-\sigma}\,k^\alpha \;-\; \alpha^2\,k^{\alpha-1}\Big] \;+\; \beta \;-\; \rho} \Rightarrow$$

$$c_c(k) = \frac{(\mu+\beta)(\mu+\rho)}{\alpha^2 Z\lambda k^{\alpha-1} - Z(\lambda-1)\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k^\alpha + \beta - \rho}[k+b(t)]$$

is the nullcline for $\dot{c} = 0$. Now, define $\bar{k}$ the level of capital such that the limit of the denominator of $c_c(k)$ goes to 0 as $k \to \bar{k}$. We know that such $\bar{k}$ exists because the denominator
$D(k) := \alpha^2 Z\lambda k^{\alpha-1} - Z(\lambda-1)\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k^{\alpha} + \beta - \rho$
has the following properties:

$$lim_{k\to 0} D(k) = +\infty$$

$$lim_{k\to +\infty} D(k) = -\infty$$

and it is continuous. Thus, there exists $\bar{k} \in [0, +\infty)$ s.th. $lim_{k\to\bar{k}} D(k) = 0$. Next, it is trivial to show that

$$lim_{k\to\bar{k}} c_c(k) = +\infty$$

since $D(k) \to 0$. It is also the case that $lim_{k\to 0} c_c(k) = 0$ since

$$lim_{k\to 0} c_c(k) = lim_{k\to 0} \frac{(\mu+\beta)(\mu+\rho)[k+b(t)]}{D(k)} = \frac{0}{+\infty} = 0$$

Also the derivative of $c_c(k)$ with respect to k is

$$\frac{dc_c(k)}{dk} = \frac{(\mu+\beta)(\mu+\rho)D(k) - D'(k)(\mu+\beta)(\mu+\rho)[k+b(t)]}{D(k)^2} > 0$$

since

$$D'(k) = (\alpha-1)\alpha^2 Z\lambda k^{\alpha-2} - \alpha Z(\lambda-1)\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k^{\alpha-1} < 0$$

So $c_c(k)$ is strictly increasing. Next, for the $\dot{k} = 0$ nullcline we have that

$$lim_{k\to 0} c_k(k) = +\infty$$

$$lim_{k\to +\infty} c_k(k) = -\infty$$

The first one follows is essentially the Inada condition while the second one is because the term with power $\alpha+1$ dominates the α, $\alpha-1$ terms as $k \Rightarrow$. Given that in the interval $[0, \bar{k}]$, $c_k(k)$ starts from $+\infty$ and ends up to $-\infty$ and $c_c(k)$ starts from 0 and asymptotically approaches $+\infty$, and both nullclines are continuous, we know there exists at least one crossing, and thus there exists a steady state.

Next, we show that there exists a $\underline{k}$ such that $\phi(\underline{k}; \sigma) = 0$ This is given by

$$\phi(\underline{k}; \sigma) = 0 \Leftrightarrow \frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}\underline{k} - \alpha^2 = 0 \Rightarrow \underline{k} = \frac{\alpha(1-\sigma)}{(1-\alpha)\eta\lambda}$$

In order for the flow rate of innovation to be positive we need two things. First, $k(0) > \underline{k}$ and second $\forall k \leq \underline{k}$, $c_k(k) > c_c(k)$. If both conditions hold, we know that $k^* \in (\underline{k}, \bar{k})$. At $k = \underline{k}$ we have

$$c_c(\underline{k}) = \frac{(\mu+\beta)(\mu+\rho)[\underline{k}+b]}{\alpha^2 Z k^{\alpha-1}}$$

$$c_k(\underline{k}) = Zk^{\alpha-1}$$

therefore we must have

$$\frac{(\mu+\beta)(\mu+\rho)[\underline{k}+b]}{\alpha^2 Z\underline{k}^{\alpha-1}} < Z\underline{k}^{\alpha-1} \Rightarrow$$

$$\frac{(\mu+\beta)(\mu+\rho)\,(\underline{k}+b)}{\alpha^2\,Z\,\underline{k}^{\alpha-1}} < Z\,\underline{k}^{\alpha-1}$$

$$\iff (\mu+\beta)(\mu+\rho)\,(\underline{k}+b) < \alpha^2\,Z^2\,\underline{k}^{2\alpha-2}$$

$$\iff \underline{k}+b < \frac{\alpha^2\,Z^2\,\underline{k}^{2\alpha-2}}{(\mu+\beta)(\mu+\rho)}$$

$$\iff b < \frac{\alpha^2\,Z^2\,\underline{k}^{2(\alpha-1)}}{(\mu+\beta)(\mu+\rho)} \;-\; \underline{k}$$

$$\iff b < \Big[\frac{\alpha^2\,Z^2}{(\mu+\beta)(\mu+\rho)\underline{k}^{3-2\alpha}} \;-1\Big] := \bar{b}$$

Therefore as long as $b < \bar{b}$ there is a steady state (k^*, c^*) such that $k^* \in (\underline{k}, \bar{k})$. The region where $\phi < 0$ is not considered. This completes the proof.

A.6 Proof of Proposition 2

Consider the system of equations (38), (39). Rewrite the equations as

$$\dot{c}(t) = \Psi^c(c,k,b) = [-Z(\lambda-1)\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k(t)^{\alpha} + Z\alpha^2\lambda k(t)^{\alpha-1} + \beta - \rho]c(t) - (\mu+\beta)(\mu+\rho)[k(t)+b(t)]$$

$$\dot{k}(t) = \Psi^k(c,k,b) = -Z(\lambda-1)\frac{\alpha(1-\alpha)}{1-\sigma}k(t)^{\alpha+1} + Z(1 - \frac{\alpha(1-\alpha)\lambda}{1-\sigma} + (\lambda-1)\alpha^2)k(t)^{\alpha} + \frac{Z\alpha^2}{\eta}k(t)^{\alpha-1} - c(t)$$

For saddle-path stability we need the determinant of the Jacobian to be negative, that is, we want

$$detJ = det\begin{pmatrix} \Psi^c_c(c^*,k^*,b) & \Psi^c_k(c^*,k^*,b) \\ \Psi^k_c(c^*,k^*,b) & \Psi^k_k(c^*,k^*,b) \end{pmatrix} < 0$$

The derivatives are given by

$$\Psi^c_c(c^*,k^*,b) = -Z(\lambda-1)\frac{\alpha(1-\alpha)\eta\lambda}{1-\sigma}k^{*\alpha} + Z\alpha^2\lambda k^{*\alpha-1} + \beta - \rho$$

$$\Psi^c_k(c^*,k^*,b) = \Big[-Z(\lambda-1)\frac{\alpha^2(1-\alpha)\,\eta\,\lambda}{1-\sigma}k^{*\alpha-1} - Z\,\alpha^2\lambda(1-\alpha)k^{*\alpha-2}\Big]\,c^* \;-\; (\mu+\beta)(\mu+\rho) < 0$$

$$\Psi^k_c(c^*,k^*,b) = -1$$

$$\Psi^k_k(c^*,k^*,b) = -Z(\lambda-1)\frac{\alpha(1-\alpha)(\alpha+1)}{1-\sigma}\,k^{*\alpha} \;+\; Z\,\alpha\Big(1 - \frac{\alpha(1-\alpha)\lambda}{1-\sigma} + (\lambda-1)\alpha^2\Big)\,k^{*\alpha-1} \;+\; \frac{Z\,\alpha^2(\alpha-1)}{\eta}\,k^{*\alpha-2}$$

So, the condition for saddle path stability is

$$\Psi^c_c(c^*,k^*,b)\Psi^k_k(c^*,k^*,b) - \Psi^c_k(c^*,k^*,b)\Psi^k_c(c^*,k^*,b) < 0 \Rightarrow$$

$$\Psi^c_c(c^*,k^*,b)\Psi^k_k(c^*,k^*,b) + \Psi^c_k(c^*,k^*,b) < 0 \Rightarrow$$

For $c^* > 0$ we require $\Psi^c_c(c^*,k^*,b) > 0$. Also, as noted above, $\Psi^c_k < 0$ So, a sufficient condition for saddle-path stability is $\Psi^k_k(c^*,k^*,b) < 0$ (which in general may not be the case hence the uncertainty about stability).

Now consider the steady state, given by c^*, k^* such that $\Psi^c = \Psi^k = 0$. From the Implicit Function Theorem we have

$$\begin{pmatrix} \partial c^*/\partial b \\ \partial k^*/\partial b \end{pmatrix} = -\frac{1}{detJ} \begin{pmatrix} \Psi_k^k & -\Psi_k^c \\ -\Psi_c^k & \Psi_c^c \end{pmatrix} \begin{pmatrix} \Psi_b^c \\ 0 \end{pmatrix} = -\frac{\Psi_b^c}{\det J} \begin{pmatrix} \Psi_k^k \\ -\Psi_c^k \end{pmatrix}$$

Therefore,

$$\partial k^*/\partial b = \frac{\Psi_b^c \Psi_c^k}{detJ}$$

Note that $\partial \Psi_b^c/\partial b < 0$ and $\Psi_c^k < 0$, hence the numerator is positive. So, if at k^*, c^* we have saddle-path stability, it must be the case that $detJ < 0$ and hence $\partial k^*/\partial b < 0$.

This, using equations (7) and (19) implies

$$\partial r^*/\partial b > 0$$

$$\partial g^*/\partial b < 0$$

Hence, as b is higher, steady state $r - g$ increases. This completes the proof.