Limit Cycles in Exchange Economies

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1. GOALS & RESULTS

Goals & Results I

- Revisit the hypothesis that business cycles are integral features of market economies even in the absence of external shocks.
- Construct a simple model of a pure exchange lifecycle economy that delivers a tent-like time map. That map has two unstable fixed points, and a single periodic attractor with period $1 + n$ for some $n \geq 1$.
- The attractor features n periods of slowly expanding credit followed by one period of abrupt reversal, as shown in Figures 1, 2, and 3.

Goals & Results II

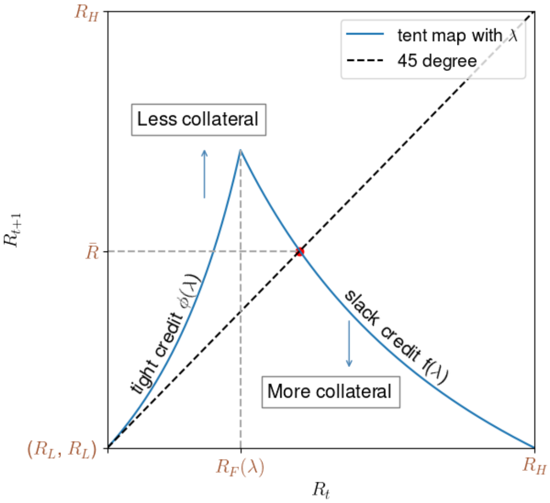

Figure 1. The role of Collateral

Goals & Results III

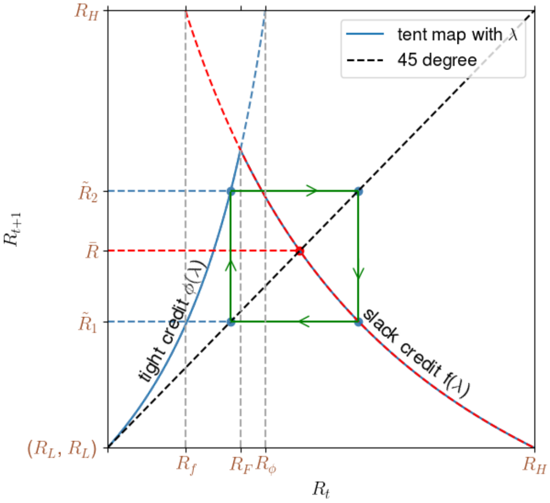

Figure 2. Limit Cycle $(\tilde{R}_1, \tilde{R}_2)$ with extreme collateral scarcity

Goals & Results IV

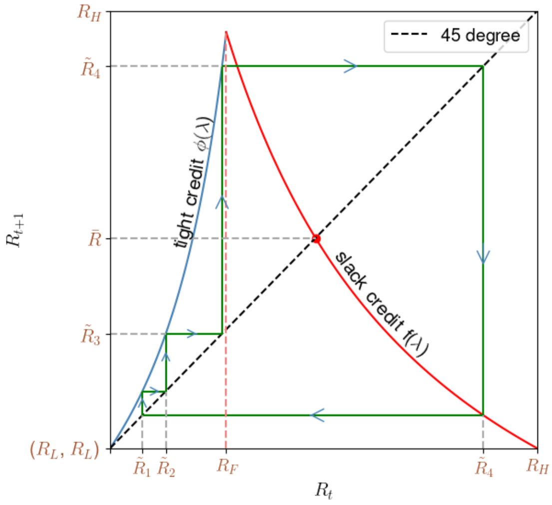

Figure 3. Limit Cycle $(\tilde{R}_1, \tilde{R}_2)$ with more extreme collateral scarcity

2. BIBLIOGRAPHY FOR LIMIT CYCLES

Bibliography for Limit Cycles I

- Keynesian literature on inventories, multipliers, and accelerators [Kaldor(1940), Samuelson(1939), Hicks(1940), Goodwin(1950)]
- Strategic complementarities in New Keynesian economies [Beaudry et al(2020)]
- Hopf cycles(or limit cycles) in multi-sector growth models [Benhabib & Nishimura (1979), Farmer & Wendner(2003)]
- Hopf cycles in two-period lifecycle growth models [Reichlin(1986)] require substitutable consumption goods plus complementary production inputs.

Bibliography for Limit Cycles II

- Imperfect credit markets amplify income shocks [Kiyotaki & Moore(1997), Kocherlakota(2000), Chakraborty(2006)]
- Lifecycle models featuring damped oscillations [Azariadis et al(2004)]

3. WHY ARE LIMIT CYCLES OUTSIDE THE MAINSTREAM?

Why are Limit Cycles outside the Mainstream? I

- Mainstream growth theory [Solow (1956)] combines simplicity, internal logic, and empirical validity. Business cycle theory (both NK macro and RBC theory) has not been as successful.
- Responses without external impulses.

4. PAPER OUTLINE

Paper Outline I

- An exchange economy with a perfect financial market converging to a unique fixed point.
- An exchange economy with financial frictions and two fixed points.
- Scarce collateral rules out convergence to either fixed points; admits asymptotically stable periodic equilibrium with period $1 + n \geq 2$, shown in Figure 3.

5. CONJECTURES

Conjectures I

- Periodicity depends on collateral scarcity.
- Extreme scarcity rules out periodic attractors; less extreme scarcity leads to a periodic attractor with $n = 1$.
- Significant scarcity raises n above one.
- Modest scarcity leads to an autarkic attractor with no borrowing or lending.

6. PERFECT FINANCIAL MARKETS

Perfect Financial Markets I

- Homogeneous cohorts $t = 0, 1, 2, \ldots$ with unit population mass.
  - 3-pd lifespan with stationary structure.
  - discounted logarithmic utility with discount factor $\beta \in [0, 1]$.
  - consumption profile $c_t = (c_t^t, c_{t+1}^t, c_{t+2}^t)$
  - endowment profile $\omega_t = (e_0, 1, e_2)$.
  - asset profile $b_t^t = (b_t^t, b_{t+1}^t, 0)$.
- Loan yields $R_t \in [R_L, R_H] := [\frac{e_2}{\beta}, \frac{1}{\beta e_0}]$.
  - middle-aged repay loans to old; save for old age.
  - lifetime budget constraints

$$c_t^t - e_0 + \frac{(c_{t+1}^t - 1)}{R_t} + \frac{(c_{t+2}^t - e_2)}{R_t R_{t+1}} \leq 0 \tag{1}$$

Perfect Financial Markets II

- debt load on cohort $t = 0, 1, \ldots$ at time $t + 1$

$$w_{t+1} = \mathcal{W}(R_t, R_{t+1}) := \sigma[1 + \beta \frac{R_L}{R_{t+1}} - (1+\beta)\frac{R_t}{R_H}] \tag{2a}$$

where

$$(\frac{1}{\sigma}, s) := (1 + \beta + \beta^2, \frac{\beta}{1+\beta}) \tag{2b}$$

- the loan market
  - demand for loans by cohort t at time t

$$\frac{w_{t+1}}{R_t} \tag{3a}$$

Perfect Financial Markets III

- cohort $t-1$ endowment profile at time t

$$(1-w_t, e_2) := (1-w_t, \beta R_L)$$

- loans supplied by cohort $t-1$ at time t

$$s(1-w_t-\frac{R_L}{R_t}) \tag{3b}$$

- loan market clears at time t

$$w_{t+1} = s[R_t(1-w_t)-R_L] \tag{4}$$

Perfect Financial Markets IV

- or equivalently

$$\mathcal{W}(R_t, R_{t+1}) = s[R_t - R_L - R_t \mathcal{W}(R_{t-1}, R_t)] \tag{5}$$

Result #1 [cf. Azariadis et al.(2004) for a general proof]:

i. given an initial distribution of financial wealth across cohorts, equation(5) has a unique fixed point $R^* \in [R_L, R_H]$
ii. the fixed point R^* in a saddle
iii. all solutions to equation(5) converge to R^* via damped oscillations

7. WHY OSCILLATE?

Why Ocillatie? I

- The ups & downs of debt loads
  - borrowing in youth & saving at middle age depend on endowment profile in middle age:

$$\tilde{w} = (1 - w, e_2)$$

  - saving at middle age:

$$\frac{1}{1+\beta}[\beta(1-w) - \frac{e_2}{R}]$$

    negatively correlated with borrowing w when young
  - cohort that borrows a lot saves little due to *consumption smoothing*
  - loans & credit fluctuate about their fixed-point values
  - smoothing consumption leads to damped oscillations

8. BORROWING CONSTRAINTS

Borrowing Constraints I

- The role of the collateral parameter λ
  - Kiyotaki-Moore constraint

    connect debt due to the value of future assets

$$w_{t+1} = -R_t b_t^t \leq \lambda b_{t+1}^t \tag{6a}$$

    for some exogenous value of the leverage ratio λ
  - equation(6a) equivalent to debt constraint

$$w_{t+1} = L(R_{t+1}) := \gamma(1 - \frac{R_L}{R_{t+1}}) \tag{6b}$$

    where

$$\gamma := \frac{\lambda\beta}{1+\beta+\lambda\beta} \in [0,1] \tag{6c}$$

Borrowing Constraints II

- Note that expectations of high future yields relax the borrowing constraint on young households (6b) because they lower the future cost of providing for old age

Result #2: If the leverage ratio λ is small enough, i.e., $\lambda \leq 1$, then the perfect market fixed point R^* cannot be an equilibrium outcome [cf. equation(5) & (6a, b, c)]

- small λ neutralizes consumption smoothing, interferes with borrowing

- Equilibrium in the loan market is now described by equation(5) plus

$$w_{t+1} = \min\{\mathcal{W}(R_t, R_{t+1}), L(R_{t+1})\} \tag{7}$$

Borrowing Constraints III

- we rule out *uniformly slack constraints* or *collateral abundance* by assuming

$$\mathcal{W}(R_t, R_{t+1}) \geq L(R_{t+1}) \tag{8}$$

for all $(\lambda, R_t, R_{t+1}) \in [0, R_L] \times [R_L, R_H]^2$

9. EQUILIBRIUM WITH SHORT COLLATERAL

Equilibrium with Short Collateral I

- For any initial volume of youthful debt, equilibrium sequences of yields $R_t \in [R_L, R_H]$ satisfy equations(4), (7), and (8), plus the collateral shortage restriction $\lambda \in [0, R_L]$
  - Typical Equilibria will switch between tight and slack collateral constraints, that is, between $\mathcal{W}(R_{t-1}R_t)$ & $L(R_t)$
  - Slack constraints require large loans that violate collateral requirements; they cannot last more than one period.

Result $\#$3: Equilibrium regimes either preserve tight constraints $[(w_t, w_{t+1}) = L(R_t), L(R_{t+1})]$; or switch from tight to slack for one period only $[\mathcal{W}(R_t, R_{t+1} = L(R_{t+1}))]$.

Equilibrium with Short Collateral II

- Short-collateral outcomes correspond to a tent-like dynamical system

$$\begin{aligned} R_{t+1} &= \phi(R_t) \qquad \text{if } R_t \in [R_L, R_F] \\ &= f(R_t) \qquad \text{if } R_t \in [R_L, R_F] \end{aligned} \tag{9}$$

where

$$\phi(R) := \frac{\lambda R_L}{\lambda + R_L - R} \tag{10a}$$

describes the tight-to-tight transition;

$$f(R) := \frac{\theta R_L R_H}{R - (1-\theta) R_H} \tag{10b}$$

Equilibrium with Short Collateral III

defines the partially unconstrained state;

$$\theta := \frac{\gamma + \beta\gamma}{(1+\beta)\sigma} \in [0, 1] \ ; \tag{10c}$$

and the critical yield R_F is defined from

$$R_F = \frac{\lambda + \theta R_L}{\lambda + \theta R_H} R_H \tag{10d}$$

- Figures 1, 2, and 3 illustrate the system (9). The important takeaway is that the entire map (ϕ, f) rises when λ drops, that is, when *collateral shrinks and financial frictions worsen*.

10. CONJECTURES REGARDING LIMIT CYCLES

Conjectures Regarding Limit Cycles I

- Conjecture 1: Two unstable fixed points exist at $R \in \{R_L, \bar{R}\}$ with $\phi'(R_L) = \frac{R_L}{\lambda} > 1$ and $f'(\bar{R}) < -1$ if λ is small & f is steep
- Conjecture 2: If λ is very small (extremely short collateral), then there may be no periodic equilibrium at all. Slightly higher λ is expected to result in a period cycle with period 2, as in Figure 2.
- Conjecture 3: Raising λ a bit more, but still keeping it below R_L, may result in an asymmetric cycle, as shown in Figure 3. Hence, n periods of expanding credit follow one period of a credit crunch.
- Conjecture 4: Raising λ slightly above R_L endows the fixed point R_L with asymptotic stability and turns it into a stationary attractor. Borrowing and lending shrink to zero.