Uniform inference with AR(p) processes

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Introduction

- CI for the coefficient in an AR(1) process uniform over the parameter space
$$\rho \in \Theta_\rho = [-M, M], \text{ some } M > 0$$
- Contributions of the paper:
  - New endogenously constructed instruments - valid inference in explosive and unit root regions
  - Combine new instruments with Phillips and Magdalinos (2009) to cover $(-\infty, \infty)$
- Combined data-driven procedure:
  - no discontinuities in the limit distribution
  - mixed normal limit distribution in all cases
  - standard inference (testing and CIs) based on $\mathcal{N}(0,1)$
  - uniform size
  - first distribution-free inference procedure in explosive case
- Extension of the methodology to AR(p) processes

Plan for the Talk

- Motivation
- Assumptions and main results: AR(1) case
- Monte Carlo simulations
- AR(p): OLS limit theory for Jordan blocks of repeated characteristic roots
- Uniform inference in AR(p) processes

Papers for the talk (Real and Imaginary parts)

1. Uniform inference with general autoregressive processes
   - AR(1) process; predictive regression; local projections (sweat, blood and tears R&R *ReStud*)
   - Drifting sequences of parameters in both AR roots and innovations
   - Background on OLS drifting sequence asymptotics on the explosive side (R&R *ET*)
2. Limit theory for AR(p) processes on the boundaries of nonstationarity
   - OLS limit theory for *near*-stationary/UR/explosive AR(p) processes: generalises Chan and Wei (1988, AoS) Jeganathan (1988, AoS)
   - Drifting sequence asymptotics necessary for establishing uniformity
3. Uniform inference in higher order autoregressive processes
   - Generalisation of 1 to AR(p) process

Assumptions: AR(1) process

AR(1) process with an intercept

$$x_t = \mu + X_t, \quad X_t = \rho X_{t-1} + u_t \tag{1}$$

Ass 1 $\rho \in \Theta_\rho := [-M, M]$ *for some* $M > 0$.

Ass 2 $(u_t, \mathcal{F}_t)_{t\in\mathbb{Z}}$ *is a martingale difference sequence such that for some* $\delta, B \in (0, \infty)$

$$\liminf_{t\to\infty} \mathbb{E}_{\mathcal{F}_{t-1}} |u_t| \geq \delta \ \ a.s. \tag{2}$$

and $\sigma_t^2 := \mathbb{E}_{\mathcal{F}_{t-1}}(u_t^2)$ *and* $\sigma^2 := \mathbb{E}\sigma_t^2$ *satisfy one of the following conditions:*

(i) $\sigma_t^2 = \sigma^2$ *a.s. for all* t *and* $(u_t^2)_{t\in\mathbb{Z}}$ *is UI sequence.*

(ii) σ_t^2 *is ARCH*(∞)*:* $u_t = \eta_t \sigma_t$, $\sigma_t^2 = \omega + \sum_{i=1}^\infty \alpha_i u_{t-i}^2$,
$\mathbb{E}(\sigma_1^4) \leq B$, $\sup_{t\geq 1} \sigma_t^2 < \infty$ *a.s.*,
$\sum_{i=1}^\infty \alpha_i \leq 1 - \delta$, $\sum_{i=1}^\infty i^\delta \alpha_i \leq B$

Ass 3 X_0 *is* $\mathcal{F}_0$*-measurable and* $\mathbb{E}|X_0|^\delta \leq B < \infty$ *for some* $\delta > 0$.

Parameter space

$\Theta_\rho = [-M, M]$, some $M > 0$
$\Theta_u = \left\{(F_t)_{t\in\mathbb{Z}} : F_t(x) = \mathbb{P}(u_t \leq x) \text{ and } (u_t)_{t\in\mathbb{Z}} \text{ satisfies Ass 2}\right\}$
$\Theta_{X_0} = \left\{F_{X_0} : F_{X_0}(x) = \mathbb{P}(X_0 \leq x) \text{ and } X_0 \text{ satisfies Ass 3}\right\}$

- Parameter space:

$$\Theta = \Theta_\rho \times \Theta_u \times \Theta_{X_0}$$

$\Theta_u^{\text{hom}} = \left\{(F_t)_{t\in\mathbb{Z}} \in \Theta_u : \sigma_t^2 = \sigma^2\right\}$

- Restriction of the parameter space under cond. homo:

$$\Theta^{\text{hom}} = \Theta_\rho \times \Theta_u^{\text{hom}} \times \Theta_{X_0}$$

Ass 4: drifting sequences of parameters

AR(1) process with an intercept

$$x_{n,t} = \mu + X_{n,t}, \quad X_{n,t} = \rho_n X_{n,t-1} + u_{n,t}$$

$\theta_n = \left(\rho_n, (F_{n,t})_{t\in\mathbb{Z}}, F_{n,X_0}\right) \in \Theta$ *satisfies the following:*

1. $n\left(|\rho_n| - 1\right) \to c \in [-\infty, \infty]$
2. $X_{n,0} \to_d X_0$ *where* X_0 *is a* $\mathcal{F}_0$*-measurable with* $\mathcal{F}_0 = \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{n,0}\right)$.

Ass 4: drifting sequences of parameters

$(u_{nt}, \mathcal{F}_{nt})_{t\in\mathbb{Z}}$ *is a martingale difference array such that*

$$\liminf_{n\to\infty} \liminf_{t\to\infty} \mathbb{E}_{\mathcal{F}_{n,t-1}} |u_{n,t}| > 0 \text{ a.s.}, \quad \sigma_n^2 := \mathbb{E}\left(\sigma_{n,t}^2\right) \to \sigma^2 > 0$$

and $\sigma_{n,t}^2 := \mathbb{E}_{\mathcal{F}_{n,t-1}}\left(u_{n,t}^2\right)$ *satisfies one of the following conditions:*

(i) $\sigma_{n,t}^2 = \sigma_n^2$ *a.s. for all* t *and* $\max_{1\le t\le n} \mathbb{E}\left(u_{n,t}^2 \mathbf{1}\left\{u_{n,t}^2 > \lambda_n\right\}\right) \to 0$ *when* $\lambda_n \to \infty$.

(ii) *For each* n, $\left(u_{n,t}, \sigma_{n,t}^2\right)_{t\in\mathbb{Z}}$ *is strictly stationary with* $\sigma_{n,t}^2 > 0$ *a.s.,*

$$\sup_{n\ge 1} \mathbb{E}\left(\sigma_{n,1}^4\right) < \infty, \quad \limsup_{n\to\infty} \sup_{t\in\mathbb{N}} \sigma_{n,t}^2 < \infty \text{ a.s.}$$

and there exist $b > 0$ $(\psi_m)_{m\in\mathbb{N}}$ *and* $(\tilde{\psi}_n)_{n\in\mathbb{N}}$ *satisfying* $\psi_m \to 0$ *and* $\tilde{\psi}_n \to 0$ *such that*

$$\sup_{t\in\mathbb{N}} \left\|\mathbb{E}_{\mathcal{F}_{n,t-1-m}}\left(\sigma_{n,t}^2 - \sigma_n^2\right)\right\|_{L_2} \le b\left(\psi_m + \tilde{\psi}_n\right) \quad \text{for all} \quad m, n \ge 1.$$

Categorisation of AR regions

- *Let*

$$c := \lim_{n \to \infty} n\left(|\rho_n| - 1\right)$$

exist in $[-\infty, \infty]$.

- Then x_t belongs to one of the following classes:

C(i) *near-stationary* processes if $c = -\infty$

C(ii) *LUR* processes if $c \in \mathbb{R}$

C(iii) *near-explosive* processes if $c = \infty$

Classes C(i)-C(iii) have *regular* and *oscillating* subclasses:
$C(i) = C_+(i) \cup C_-(i)$, etc

Lemma

(i) *Let* $(\rho_n)_{n\in\mathbb{N}}$ *satisfy* $\rho_n \to \rho \in \mathbb{R}$. *For any subsequence* $\left(\rho_{m_n}\right)_{n\in\mathbb{N}}$ *of* $(\rho_n)_{n\in\mathbb{N}}$ *there exists a further subsequence* $\left(\rho_{s_n}\right)_{n\in\mathbb{N}}$ *such that*

$$s_n\left(\left|\rho_{s_n}\right| - 1\right) \to c \in [-\infty, \infty]$$

(ii) *Let* $(u_{n,t})_{t\in\mathbb{N}}$ *be an ARCH*(∞) *satisfying Assumption 2(ii). For any subsequence* $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ *there exists a further subsequence* $(k_n)_{n\in\mathbb{N}} \subseteq (m_n)_{n\in\mathbb{N}}$ *such that* $\sigma^2_{k_n,t} = \mathbb{E}_{\mathcal{F}_{k_n,t-1}} u^2_{k_n,t}$ *satisfies*

$$\sup_{t\in\mathbb{N}} \left\| \mathbb{E}_{\mathcal{F}_{k_n,t-1-m}} \left(\sigma^2_{k_n,t} - \sigma^2_{k_n}\right) \right\|_{L_2} \leq b\left(\psi_m + \tilde{\psi}_{k_n}\right)$$

where $\psi_m \to 0$ *and* $\tilde{\psi}_n \to 0$.

Lemma (useful in the explosive case)

For each $t \in \mathbb{Z}$ *let* $u_t := \liminf_{n\to\infty} u_{n,t}$ *and*

$$\mathcal{G}_{n,t} := \cap_{j=n}^{\infty} \mathcal{F}_{j,t} \text{ and } \mathcal{F}_t := \sigma\left(\cup_{n=1}^{\infty} \mathcal{G}_{n,t}\right) = \sigma\left(\liminf_{n\to\infty} \mathcal{F}_{n,t}\right).$$

(i) *The sequence* $(u_t, \mathcal{F}_t)_{t\in\mathbb{Z}}$ *is a martingale difference satisfying the local MZ conditions*

$$\liminf_{t\to\infty} \mathbb{E}\left(|u_t| \,|\, \mathcal{F}_{t-1}\right) > 0 \text{ a.s. and } \sup_{t\in\mathbb{N}} \mathbb{E}\left(u_t^2 \,\middle|\, \mathcal{F}_{t-1}\right) < \infty \text{ a.s.}$$

(ii) *Let*

$$\xi_n = \left(\rho_n^2 - 1\right)^{1/2} \sum_{t=1}^{n} \rho_n^{-t} u_{n,t}.$$

For any subsequence of $(\xi_n)_{n\in\mathbb{N}}$ *there exists a further subsequence that converges in distribution. If* $\xi_{k_n} \to_d \xi_\infty$ *for some subsequence* $(\xi_{k_n})_{n\in\mathbb{N}}$ *of* $(\xi_n)_{n\in\mathbb{N}}$*, then* $\mathbb{P}(\xi_\infty = 0) = 0$.

Combined instrument: robust inference in all regions

$$F_n = \{n(|\hat{\rho}_n| - 1) \leq 0\}$$
$$F_n^+ = F_n \cap \{\hat{\rho}_n \geq 0\} \qquad F_n^- = F_n \cap \{\hat{\rho}_n < 0\}$$
$$\bar{F}_n^+ = \bar{F}_n \cap \{\hat{\rho}_n \geq 0\} \qquad \bar{F}_n^- = \bar{F}_n \cap \{\hat{\rho}_n < 0\}$$

- Construct a combined instrument:

$$z_t = \rho_{nz} z_{t-1} + \tilde{u}_t$$

$$\rho_{nz} = \varphi_{1n}\mathbf{1}_{F_n^+} + \varphi_{1n}^-\mathbf{1}_{F_n^-} + \varphi_{2n}\mathbf{1}_{\bar{F}_n^+} + \varphi_{2n}^-\mathbf{1}_{\bar{F}_n^-} \qquad \varphi_{1n}, \varphi_{2n} \to 1$$
$$\tilde{u}_t = \Delta x_t \mathbf{1}_{F_n^+} + \nabla x_t \mathbf{1}_{F_n^-} + \hat{u}_t \mathbf{1}_{\bar{F}_n^+ \cup \bar{F}_n^-} \qquad \varphi_{1n}^-, \varphi_{2n}^- \to -1$$

$\varphi_{1n} \in C_+(i), \varphi_{1n}^- \in C_-(i), \varphi_{2n} \in C_+(iii), \varphi_{2n}^- \in C_-(iii)$

- Since $\mathbf{1}\{F_n\} \to_p 1$ in C(i), a NSt instrument is chosen
- Since $\mathbf{1}\{F_n\} \to_p 0$ in C(iii), a NExp instrument is chosen
- In the LUR region, we show z_t: combination of the two instruments that **preserves the asymptotic mixed Gaussianity** of the IV estimator

IV estimator and t-statistic

- IV estimator $\tilde{\rho}_n = \sum_{t=1}^{n} x_t z_{t-1} / \sum_{t=1}^{n} x_{t-1} z_{t-1}$
- t-statistic and CI under $\theta \in \Theta^{\text{hom}} = \Theta_\rho \times \Theta_u^{\text{hom}} \times \Theta_{X_0}$:

$$\tilde{T}_n(\rho) = \sqrt{\sum_{t=1}^{n} x_{t-1} z_{t-1} / \hat{\sigma}_u^2 \sum_{t=1}^{n} z_{t-1}^2}\,(\tilde{\rho}_n - \rho_n)$$

$$I_n(\rho, \alpha) = [\tilde{\rho}_n - c_n(\alpha), \tilde{\rho}_n + c_n(\alpha)]$$

$$c_n(\alpha) = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \hat{\sigma}_u / \sqrt{\sum_{t=1}^{n} x_{t-1} z_{t-1} / \sum_{t=1}^{n} z_{t-1}^2}$$

- t-statistic and CI under $\theta \in \Theta = \Theta_\rho \times \Theta_u \times \Theta_{X_0}$:

$$T_n^{EW} = Q_n^{-1/2}(\tilde{\rho}_n - \rho)$$

$$Q_n = \left(\underline{X}' \tilde{Z}\right)^{-2} \sum_{t=1}^{n} \tilde{z}_{n,t-1}^2 \left(\hat{u}_{n,t}^2 \mathbf{1}_{F_n} + \hat{\sigma}_n^2 \mathbf{1}_{\bar{F}_n}\right)$$

Theorem: drifting sequences

$$\Theta_0 = \{\theta \in \Theta : F_t(x) = F_{\sigma^2}(x) \quad \forall t \in \mathbb{Z}\}$$

- Normalisation:

$$\pi_n = \begin{cases} n^{1/2}\left(1-\rho_n^2\phi_{1n}^2\right)^{-1/2}, & \text{C(i)} \\ n^{1/2}\left(1-\phi_{1n}^2\right)^{-1/2}\mathbf{1}_{F_n} + 2n^{1/2}\left(\phi_{2n}^2-1\right)^{-1/2}\mathbf{1}_{\bar{F}_n}, & \text{C(ii)} \\ \left(\phi_{2n}^2-1\right)^{1/2}\left(|\rho_n|\,|\phi_{2n}|-1\right)^{-1}\left(\rho_n^2-1\right)^{-1/2}|\rho_n|^n, & \text{C(iii)} \end{cases}$$

- *For arbitrary sequences* $(\theta_n)_{n\in\mathbb{N}}$ *in* Θ *and* $(\theta'_n)_{n\in\mathbb{N}}$ *in* Θ^{hom} *satisfying Assumption 4:*

(i) $\lim_{n\to\infty}\mathbb{P}_{\theta_n}(\pi_n(\tilde{\rho}_n-\rho_n)\le x) = \mathbb{P}_{\theta_0}(\mathcal{L}\le x)$, $\mathcal{L} =_d \mathcal{MN}(0,V)$

(ii) $\lim_{n\to\infty}\mathbb{P}_{\theta'_n}(|T_n|\le x) = \Phi(x)$

(iii) $\lim_{n\to\infty}\mathbb{P}_{\theta_n}(|T_n^{EW}|\le x) = \Phi(x)$

Theorem: uniform inference AR(1)

- CR and CI under cond. homo:

$$\mathcal{R}_{n,\alpha} = \left\{ |T_n| > \Phi^{-1}(1-\alpha/2) \right\} \text{ and } I_n(\tilde{\rho}_n, \alpha)$$

- CR and CI under cond. hetero:

$$\mathcal{R}^{EW}_{n,\alpha} = \left\{ \left| T^{EW}_n \right| > \Phi^{-1}(1-\alpha/2) \right\} \text{ and } I^{EW}_n(\tilde{\rho}_n, \alpha)$$

Theorem

For any $\alpha \in (0,1)$*:*
(i) $\lim_{n\to\infty} \sup_{\theta\in\Theta} \mathbb{P}_\theta\left(\mathcal{R}^{EW}_{n,\alpha}\right) = \alpha$ *and*
$\lim_{n\to\infty} \sup_{\theta\in\Theta^{\text{hom}}} \mathbb{P}_\theta\left(\mathcal{R}_{n,\alpha}\right) = \alpha$
(ii) $\lim_{n\to\infty} \inf_{\theta\in\Theta} \mathbb{P}_\theta\left[\rho \in I^{EW}_n(\tilde{\rho}_n, \alpha)\right] = 1-\alpha$ *and*
$\lim_{n\to\infty} \inf_{\theta\in\Theta^{\text{hom}}} \mathbb{P}_\theta\left[\rho \in I_n(\tilde{\rho}_n, \alpha)\right] = 1-\alpha.$

Instrument Selection

- Any values of φ_{1n} and φ_{2n} s.t. $\varphi_{1n} = 1 - 1/\kappa_{1n}$, $\varphi_{2n} = 1 + 1/\kappa_{2n}$ with $\kappa_{1n}, \kappa_{2n} \to \infty$, $\kappa_{1n}/n \to 0$, $\kappa_{2n}/n \to 0$: valid asymptotic inference
- But finite sample performance may vary considerably
- Choosing $\varphi_{1n} = 1 - 1/n^{b_1}$, $\varphi_{2n} = 1 + 1/n^{b_2}$: find values for b_1 and b_2 in $(0, 1)$
- We require instrument with good performance along ALL autoregressive regions without *a priori* knowledge
- We adopt a conservative approach: our inference procedure suffers the worst finite sample distortion when $\rho_n = 1$ with large correlation $\rho_{\varepsilon u}$
- We minimise the worst case scenario and select values that deliver satisfactory test size
- Grid of values for b_1 and b_2 with $|\rho_{\varepsilon u}| = 0.99$ $\Rightarrow$ $b_1 = 0.85$ and $b_2 = 0.7$

Conclusion

- New IV procedure for AR(1) regressor with arbitrary stochastic properties: combines the IVX instrument of PM(2009) for $\rho \leq 1$ with a new instrument valid for $\rho \geq 1$.
- Establishes the uniformity of tests and CIs over an AR parameter space $\Theta = [-M, M]$, some $M > 0$.
- Proves the uniformity of PM(2009) over $[-1 + \delta, 1]$
- The resulting data-driven procedure:
  - no discontinuities in the limit distribution
  - mixed normal limit distribution in all cases
  - standard inference (testing and CIs) based on $\mathcal{N}(0, 1)$
  - uniform size properties
  - invariant to the innovations' distribution(s) even in the explosive case.
- Extensions: AR(p); VAR(p); time-varying persistence/volatility

AR(p) process

$$x_t = \phi_1 x_{t-1} + ... + \phi_p x_{t-p} + u_t$$

- Roots of the characteristic polynomial $sp(R) = \{\rho_1, ..., \rho_p\}$.
- Discontinuity in the rate of convergence and the asymptotic distribution of the OLS estimator of $\phi = (\phi_1, ..., \phi_p)$ according to the properties of $sp(R)$.
- Uniform inference on ϕ requires limit distribution theory along drifting sequences

$$(\rho_{1n}, ..., \rho_{pn}) \to (\rho_1, ..., \rho_p) \tag{3}$$

- Adaptation of MP(2022) is relatively straightforward when the eigenvalues of R are *distinct*: AR(p) $\simeq$ p-AR(1) components
- R is diagonalisable *iff* it has distinct eigenvalues
- Repeated eigenvalues: possibly higher integration orders

AR(2) with distinct characteristic roots

- $\underline{x}_t = R\underline{x}_{t-1} + \underline{u}_t$
- R diagonalisable: $X_t = T_n^{-1}\underline{x}_t$

$$T_n^{-1} = \begin{bmatrix} 1 & -\rho_{2n} \\ -1 & \rho_{1n} \end{bmatrix} = \begin{bmatrix} S_{1n}' \\ S_{2n}' \end{bmatrix}$$

implies that

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} \rho_{1n} & 0 \\ 0 & \rho_{2n} \end{bmatrix} \begin{bmatrix} X_{1t-1} \\ X_{2t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ -u_t \end{bmatrix}$$

- MP(2022) instrumentation:

$$\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} = \begin{bmatrix} \varphi_{1n} & 0 \\ 0 & \varphi_{2n} \end{bmatrix} \begin{bmatrix} Z_{1t-1} \\ Z_{2t-1} \end{bmatrix} + \begin{bmatrix} \tilde{u}_{1t} \\ -\tilde{u}_{2t} \end{bmatrix} \tag{4}$$

  - $n\left(|\hat{\rho}_{in}| - 1\right) \leq 0 \Longrightarrow \varphi_{in}$ near-stationary and $\tilde{u}_{it} = \Delta x_t$
  - $n\left(|\hat{\rho}_{in}| - 1\right) > 0 \Longrightarrow \varphi_{in}$ near-explosive and $\tilde{u}_{it} = \hat{u}_t$
- Feasibility: need to estimate the *directions* S_{1n} and S_{2n}

AR(2) with distinct characteristic roots

- Estimate the *directions*

$$X_{1t} \simeq \hat{S}_{1n}' \underline{x}_t \quad \text{and} \quad X_{2t} \simeq \hat{S}_{2n}' \underline{x}_t$$

where

$$\hat{T}_n^{-1} = \begin{bmatrix} 1 & -\hat{\rho}_{2n} \\ -1 & \hat{\rho}_{1n} \end{bmatrix} = \begin{bmatrix} \hat{S}_{1n}' \\ \hat{S}_{2n}' \end{bmatrix}$$

- IV residuals:

$$\begin{aligned} \tilde{u}_{1t} &= \hat{S}_{1n}' \Delta \underline{x}_t \mathbf{1}_{F_{1n}^+} + \hat{S}_{1n}' \nabla \underline{x}_t \mathbf{1}_{F_{1n}^-} + \hat{u}_t \mathbf{1}_{\bar{F}_{1n}^+ \cup \bar{F}_{1n}^-} \\ \tilde{u}_{2t} &= \hat{S}_{2n}' \Delta \underline{x}_t \mathbf{1}_{F_{2n}^+} + \hat{S}_{2n}' \nabla \underline{x}_t \mathbf{1}_{F_{2n}^-} + \hat{u}_t \mathbf{1}_{\bar{F}_{2n}^+ \cup \bar{F}_{2n}^-} \end{aligned}$$

- Eigenvector estimation error $\hat{S}_{in}$ is controllable because it only depends on the estimation error of $\hat{\rho}_{in}$

- IV residuals:

$$\begin{aligned}\tilde{u}_{1t} &= \Delta X_{1t}\mathbf{1}_{F_{1n}^{+}} + \nabla X_{1t}\mathbf{1}_{F_{1n}^{-}} + \hat{u}_t\mathbf{1}_{\bar{F}_{1n}^{+}\cup\bar{F}_{1n}^{-}} - r_{1t}^{(n)} \\ \tilde{u}_{2t} &= \Delta X_{2t}\mathbf{1}_{F_{2n}^{+}} + \nabla X_{2t}\mathbf{1}_{F_{2n}^{-}} - \hat{u}_t\mathbf{1}_{\bar{F}_{2n}^{+}\cup\bar{F}_{2n}^{-}} + r_{2t}^{(n)}\end{aligned}$$

  where

$$r_{1t}^{(n)} = (\hat{\rho}_{2n} - \rho_{2n})\left(\Delta x_{t-1}\mathbf{1}_{F_{1n}^{+}} + \nabla x_{t-1}\mathbf{1}_{F_{1n}^{-}}\right) \quad (5)$$

$$r_{2t}^{(n)} = (\hat{\rho}_{1n} - \rho_{1n})\left(\Delta x_{t-1}\mathbf{1}_{F_{2n}^{+}} + \nabla x_{t-1}\mathbf{1}_{F_{2n}^{-}}\right) \quad (6)$$

- If ρ_{2n} is explosive and ρ_{1n} is stable, x_t is explosive, but:
  - the exponential decay of $\hat{\rho}_{2n} - \rho_{2n}$ controls (5)
  - the arbitrary decay of $\mathbf{1}_{F_{2n}^{+}}$ and $\mathbf{1}_{F_{2n}^{-}}$ controls (6) (Lemma 2 of MP(2022)).
- The above ensures the feasibility of the instrumentation in (4).

IV estimator (distinct characteristic roots)

- IV estimator $\tilde{\phi}_n$ of ϕ: $\tilde{\phi}_n = \left(\sum_{t=1}^{n} \underline{z}_{t-1}\underline{x}'_{t-1}\right)^{-1} \sum_{t=1}^{n} \underline{z}_{t-1} x_t$

$$T_n = (\rho_{1n} - \rho_{2n})^{-1} \begin{bmatrix} \rho_{1n} & \rho_{2n} \\ 1 & 1 \end{bmatrix}.$$

- Normalisation D_n depends on the AR regime of ρ_{1n}, ρ_{2n} and their relative order to the instrument Z_t:

$$D_n T_n' (\tilde{\phi}_n - \phi) = \left(D_{nz}^{-1} \sum_{t=1}^{n} Z_{t-1} X_{t-1}' D_n^{-1}\right)^{-1} D_{nz}^{-1} \sum_{t=1}^{n} Z_{t-1} u_t$$

Theorem

The limit distribution of $\tilde{\phi}_n$ is given by

$$D_n T_n' (\tilde{\phi}_n - \phi) \to_d \mathcal{MN}\left(0, \sigma^2 \left(V_{xz}^{-1}\right)' V_{zz} V_{xz}^{-1}\right)$$

where $V_{zz} > 0$ and $\det(V_{xz}) \neq 0$ a.s.

AR(2) with repeated characteristic root

- ρ_n repeated characteristic root

$$J(\rho_n) = \begin{bmatrix} \rho_n & 0 \\ 1 & \rho_n \end{bmatrix}, \; T^{-1}(\rho_n) = \begin{bmatrix} \rho_n & -\rho_n^2 \\ 1 & 0 \end{bmatrix}$$

and $X_t := T^{-1}(\rho_n)\, \underline{x}_t$, the Jordan decomposition gives

$$X_t = J(\rho_n)\, X_{t-1} + U_t, \quad U_t = u_t \begin{bmatrix} \rho_n \\ 1 \end{bmatrix} \tag{7}$$

- Instrument (7) by

$$Z_t = J(\varphi_n)\, Z_{t-1} + \tilde{u}_t \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix} \tag{8}$$

  - $F_n = \{n(|\hat{\rho}_n| - 1) \leq 0\} \Rightarrow \varphi_n$ near-stationary, $\tilde{u}_t = \Delta^2 x_t$
  - $\bar{F}_n = n(|\hat{\rho}_n| - 1) > 0 \Rightarrow \varphi_n$ near-explosive, $\tilde{u}_t = \hat{u}_t$

Estimation of Jordan form

- Data driven criterion for "equality" of ρ_{1n} and ρ_{2n}
- Estimation of the Jordan form of the companion matrix
- Useful for building the instrument process
- Use Pitman drift as a measure of proximity of ρ_{1n} to ρ_{2n}
- It is easy to see

$$\hat{\rho}_{1n} - \hat{\rho}_{2n} = \rho_{1n} - \rho_{2n} + O_p\left[n^{-1/2}\left(|\rho_{1n} - 1|^{1/2} \vee |\rho_{2n} - 1|^{1/2}\right)\right]$$

- Define the sequence

$$\begin{aligned} l_n &= n^{1/2}\left(|\hat{\rho}_{1n} - 1|^{-1/2} \wedge |\hat{\rho}_{2n} - 1|^{-1/2}\right)/L_n \qquad (9) \\ &\asymp n^{1/2}\left(|\rho_{1n} - 1|^{-1/2} \wedge |\rho_{2n} - 1|^{-1/2}\right)/L_n \end{aligned}$$

  for some $L_n \to \infty$
- Use the following criterion for (4) vs. (8) instrumentation:

$$\rho_{1n} \simeq \rho_{2n} \quad \text{if} \quad l_n\,|\hat{\rho}_{1n} - \hat{\rho}_{2n}| < b$$

Estimation of Jordan form

- Estimated form of the companion matrix
- Define the sequence of events

$$G_n := \{ I_n \, |\hat{\rho}_{1n} - \hat{\rho}_{2n}| < b \}$$

- Instrument process:

$$Z_t = J_n Z_{t-1} + v_t$$

where

$$\begin{aligned} J_n &= \mathbf{1}_{G_n} \begin{bmatrix} \varphi_n & 0 \\ 1 & \varphi_n \end{bmatrix} + \mathbf{1}_{\bar{G}_n} \begin{bmatrix} \varphi_{1n} & 0 \\ 0 & \varphi_{2n} \end{bmatrix} \\ v_t &= \mathbf{1}_{G_n} \tilde{u}_t \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix} + \mathbf{1}_{\bar{G}_n} \begin{bmatrix} \tilde{u}_{1t} \\ -\tilde{u}_{2t} \end{bmatrix} \end{aligned}$$

OLS limit theory with near-stationary repeated ch. root

- Fixed parameter AR(p): Chan and Wei (1988, AoS)
- Extension to $\rho_n \longrightarrow 1$ in C(i):

$$D(\rho_n) = diag\left[\left(1-\rho_n^2\right)^{-1/2}, \left(1-\rho_n^2\right)^{-3/2}\right]$$

$$D(\rho_n)^{-1} \frac{1}{n}\sum_{t=1}^{n} X_{t-1}X_{t-1}' D(\rho_n)^{-1} \longrightarrow_p \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} > 0 \tag{10}$$

- Since $Z_t \simeq X_t$ when $(\rho_n - 1)/(\varphi_n - 1) \longrightarrow \infty$ and Z_t behaves asymptotically as

$$Z_t = J(\varphi_n) Z_{t-1} + u_t \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix} \text{ when } (\rho_n - 1)/(\varphi_n - 1) \longrightarrow 0$$

(10) with $(D(\rho_n), X_{t-1})$ replaced by $(D(\varphi_n), Z_{t-1})$ establishes asymptotic instrument validity.

OLS limit theory with near-explosive repeated ch. root

- Fixed explosive parameter AR(p): Jeganathan (1988, AoS)
- Extension to $\rho_n \longrightarrow 1$ in C(iii):

$$\left(\rho_n^2 - 1\right) D\left(\rho_n\right)^{-1} J\left(\rho_n\right)^{-n} \sum_{t=1}^{n} X_{t-1} X_{t-1}' J'\left(\rho_n\right)^{-n} D\left(\rho_n\right)^{-1} \rightarrow_d M\left(\xi_1, \xi_2\right) \tag{11}$$

$$M\left(\xi_1, \xi_2\right) = \begin{bmatrix} \xi_1^2 & \xi_1 \xi_2 \\ \xi_1 \xi_2 & \xi_1^2 + \xi_2^2 \end{bmatrix} > 0 \quad \left(\xi_1, \xi_2\right)' =_d \mathcal{N}\left(0, \sigma^2 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\right)$$

- Since the instrument process Z_t behaves asymptotically as

$$Z_t = J\left(\varphi_n\right) Z_{t-1} + u_t \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix}$$

(11) with $\left(D\left(\rho_n\right), X_{t-1}\right)$ replaced by $\left(D\left(\varphi_n\right), Z_{t-1}\right)$ establishes asymptotic instrument validity.

Signal matrix summary

- Key result:

$$D_n^{-1}\frac{1}{n}\sum_{t=1}^{n}X_{t-1}Z_{t-1}'D\left(\varphi_n\right)^{-1}\rightarrow_d\Psi,\ \det\left(\Psi\right)\neq 0\ a.s.$$

  where

$$D\left(\varphi_n\right)=diag\left[\left|1-\varphi_n^2\right|^{-1/2},\left|1-\varphi_n^2\right|^{-3/2}\right]$$

  a stochastic normalisation matrix D_n satisfying

$$\|D_n\|\rightarrow\infty,\ \ D_n\in m\sigma\left(B\left(t\right):t\in[0,1]\right),\ \ n^{-1/2}\sum_{j=1}^{\lfloor nt\rfloor}u_j\Rightarrow B\left(t\right)$$

- $n^{-1/2}D\left(\varphi_n\right)^{-1}\sum_{t=1}^{n}Z_{t-1}u_t$ and $\sigma\left(B\left(t\right):t\in[0,1]\right)$ asymptotically independent.
- Under the above conditions the IV-Wald statistic for testing q restrictions will be asymptotically $\chi^2\left(q\right)$ and the associated CR will have uniform size properties as in the AR(1) case.