Optimal Taxation with Heterogeneous Agents

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Motivation

- Should governments adjust the time path of taxes in response to inequality?
  - Rapidly growing literature investigates the properties of optimal Ramsey policies in HA models.
  - Bhandari, Golosov Sargent Evans, Auclert, Straub, Ronglie, Ragot, Hagedorn, Holter and many others...
  - ... mostly focus on MIT shocks (assuming linearity in aggregate state variables) or consider the properties of the long run steady state.
- This paper: Solves for optimal fiscal policies in a full blown stochastic model, extending the seminar paper of Aiyagari, Marcet, Sargent and Sepalla (2002) to HA.

Motivation

- Why? Broad motivation similar to recent literature. Heterogeneity is realistic, and we want to know whether it matters for government policy. But considering fully stochastic equilibria could also be important.
  - Properties of stochastic steady state could be quite different. (e.g. AMSS precautionary savings accumulations by the optimizing government).
  - Optimal taxation depends on the time varying level of government debt (Lucas and Stockey (1983), AMSS, Faraglia et al (2016) and others).
    - In addition it may also depend on inequality/ Intertemporal tax distortions driven by redistribution.
  - Distributions of assets and promises may be important state variables. (Assuming linearity in aggregate states not wlog).
  - When debt is long term, in principle, many lags of these distributions can matter for the solution...

- This paper

  - Develops an algorithm to solve this class of models with large state spaces.

  - Key finding is 'approximate aggregate' holds for a large class of models. The means of the distributions are sufficient state variables, we do not need higher order moments.

  - Tax schedule driven by average debt levels, not by redistribution. Inequality matters only in the initial allocation (and is a source of time inconsistency).

  - Average debt level has a negative trend (precautionary savings by the government).
  - These results extend to models with long term debt...

Baseline Model

We consider a competitive equilibrium of an economy with $N \geq 2$ heterogeneous agents. Time is discrete $t \in \{0, 1, 2, ..., \infty\}$. Each agent derives utility

$$\mathbf{E}_0 \sum_{t \geq 0} \beta^t \widetilde{u}(c_t^i, h_t^i)$$

where c_t^i denotes the consumption of agent i and h_t^i is hours worked.

Agents maximize utility subject to

$$q_t a_t^i = a_{t-1}^i - c_t^i + h_t^i \epsilon_t^i (1 - \tau_t)$$

q_t = bond price. ϵ_t^i = idiosyncratic productivity (standard Markov process). τ_t = labour tax.

Baseline Model

Assume separable utility. Optimality gives

$$v_{h,t}^{i} = (1-\tau_t)\epsilon_t^i u_{c,t}^i \qquad \text{and} \qquad q_t = \beta \mathbf{E}_t \frac{u_{c,t+1}^i}{u_{c,t}^i}$$

and the transersality condition

$$\lim_{j\to\infty} q_{t+j} a_{t+j}^i = 0$$

for every history s^t.

Baseline Model

The government levies distortionary taxes τ_t and issues debt b_t to finance exogenous spending g_t. The budget constraint is given by:

$$q_t b_t = b_{t-1} + g_t - \tau_t \sum_{i \in \mathbf{N}} h_t^i \epsilon_t^i$$

In equilibrium we have $b_t = \sum_{i \in \mathbf{N}} a_t^i$

The resource constraint of the economy is

$$\sum_{i \in \mathbf{N}} c_t^i + g_t = \sum_{i \in \mathbf{N}} \epsilon_t^i h_t^i$$

where $\mathbf{N} := \{1, 2, ..., N\}$.

Baseline Model: Sufficient Implementability conditions.

Lemma 1. A stochastic sequence $\left\{c_t^i, h_t^i, a_t^i\right\}_{t,i}$ and (for aggregate prices) $\left\{q_t, \tau_t\right\}_t$ is a competitive equilibrium iff it satisfies for all t, i the following conditions:

$$\sum_{i \in \mathbf{N}} c_t^i + g_t = \sum_{i \in \mathbf{N}} \epsilon_t^i h_t^i$$

$$a_{t-1}^i u_{c,t}^i = \mathbf{E}_t \sum_{j=0}^{\infty} \beta^j \left(c_{t+j}^i u_{c,t+j}^i - (1 - \tau_{t+j}) \epsilon_{t+j}^i h_{t+j}^i u_{c,t+j}^i \right)$$

$$v_{h,t}^i = (1 - \tau_t) u_{c,t}^i \epsilon_t^i$$

$$u_{c,t}^i q_t = \beta \mathbf{E}_t u_{c,t+1}^i$$

Baseline Model: Ramsey Policy

Let $\widetilde{h}^i_t(\tau_t, c^i_t) := (v^i_{h,t})^{-1}((1-\tau_t)\epsilon^i_t u^i_{c,t})$. The Lagrangean for the Ramsey program can be written as:

$$\mathcal{L} = \mathbf{E}_0 \sum_{t\geq 0} \beta^t \Bigg\{ \sum_{i\in\mathbf{N}} \Bigg(u(c^i_t) - v(\widetilde{h}^i_t) \Bigg) + \lambda^i_t \Bigg(a^i_t \beta \mathbf{E}_t u^i_{c,t+1} - a^i_{t-1} u^i_{c,t} + c^i_t u^i_{c,t} - (1-\tau_t)\widetilde{h}^i_t \epsilon^i_t u^i_{c,t} \Bigg)$$
$$-\xi_t \Bigg(\sum_{i\in\mathbf{N}} c^i_t + g_t - \sum_{i\in\mathbf{N}} \epsilon^i_t \widetilde{h}^i_t \Bigg) + \psi^i_t \Bigg(q_t u^i_{c,t} - \beta \mathbf{E}_t u^i_{c,t+1} \Bigg) \Bigg\}$$

Baseline Model: Ramsey Policy

The first order conditions characterizing the optimum are given by:

$$u^i_{c,t} - v_{\widetilde{h}^i_t}\frac{d\widetilde{h}^i_t}{dc^i_t} - a^i_{t-1}u^i_{cc,t}\left(\lambda^i_t - \lambda^i_{t-1}\right) + \lambda^i_t\Bigg(u^i_{c,t} + c^i_t u^i_{cc,t} - (1-\tau_t)\epsilon^i_t\widetilde{h}^i_t u^i_{cc,t}$$

$$-(1-\tau_t)\epsilon^i_t\frac{d\widetilde{h}^i_t}{dc^i_t}u^i_{c,t}\Bigg) - \xi_t(1-\frac{d\widetilde{h}^i_t}{dc^i_t}) + u^i_{cc,t}\left(\psi^i_t q_t - \psi^i_{t-1}\right) = 0 \quad \text{for} \quad i \in \mathbf{N} \tag{1}$$

$$-\sum_{i\in\mathbf{N}} v_{\widetilde{h}^i_t}\frac{d\widetilde{h}^i_t}{d\tau_t} + \sum_{i\in\mathbf{N}}\left[\lambda^i_t\widetilde{h}^i_t\epsilon^i_t u^i_{c,t} - (1-\tau_t)\lambda^i_t\epsilon^i_t\frac{d\widetilde{h}^i_t}{d\tau_t}u^i_{c,t}\right] + \xi_t\sum_{i\in\mathbf{N}}\frac{d\widetilde{h}^i_t}{d\tau_t}\epsilon^i_t = 0 \tag{2}$$

$$\lambda^i_t = \frac{\mathbf{E}_t\lambda^i_{t+1}u^i_{c,t+1}}{\mathbf{E}_t u^i_{c,t+1}} \quad \text{for} \quad i \in \mathbf{N} \tag{3}$$

$$\sum_{i\in\mathbf{N}}\psi^i_t u^i_{c,t} = 0 \tag{4}$$

with $\psi^i_{-1} = \lambda^i_{-1} = 0$ and given the initial distribution of assets.

Baseline Model: Ramsey Policy

(1) is FONC wrt c

$$F_t - a^i_{t-1} u^i_{cc,t}\left(\lambda^i_t - \lambda^i_{t-1}\right) + u^i_{cc,t}\left(\psi^i_t q_t - \psi^i_{t-1}\right) = 0 \quad \text{for} \quad i \in \mathbf{N}$$

$a^i_{t-1} u^i_{cc,t}\left(\lambda^i_t - \lambda^i_{t-1}\right)$ tax/interest rate manipulation (LS, AMSS, Faraglia et al (2016)).
Following a shock in g_t or ϵ^i_t distort taxes intertemporally to relax the constraint:

$$a^i_{t-1} u^i_{c,t} = \mathbf{E}_t \sum_{j=0}^{\infty} \beta^j \left(c^i_{t+j} u^i_{c,t+j} - (1 - \tau_{t+j}) h^i_{t+j} \epsilon^i_{t+j} u^i_{c,t+j} \right)$$

Term $u^i_{cc,t}\left(\psi^i_t q_t - \psi^i_{t-1}\right)$: Intertemporal tax distortions must be consistent with the requirement that the Euler constraints are satisfied.

Baseline Model: Ramsey Policy

$$\lambda_t^i = \frac{\mathbf{E}_t \lambda_{t+1}^i u_{c,t+1}^i}{\mathbf{E}_t u_{c,t+1}^i} \quad \text{for} \quad i \in \mathbf{N}$$

The Lagrange multiplier on the agent i's budget constraint evolves as a risk adjusted random walk (with risk adjustment factor $\frac{u_{c,t+1}^i}{\mathbf{E}_t u_{c,t+1}^i}$ a result which is standard in models of optimal policy under incomplete markets (AMSS).

Finally,

$$\sum_{i \in \mathbf{N}} \psi_t^i u_{c,t}^i = 0$$

determines the optimal bond price q.

Baseline Model: Ramsey Policy

Lemma 2 (State Vector). *Consider the solution of system of FONC for the idiosyncratic variables* $y_t^i := \{a_t^i, c_t^i, h_t^i, \lambda_t^i, \psi_t^i\}$ *and the aggregate variables* $x_t = \{q_t, \tau_t, \xi_t\}$. *The solution satisfies the recursive structure*

$$y_t^i = F^y(g_t, \epsilon_t^i, a_{t-1}^i, \zeta_{t-1}^i, \Gamma_{t-1})$$

$$x_t = F^x(g_t, \Gamma_{t-1})$$

$$\Gamma_t = T(\Gamma_{t-1}, g_t, g_{t-1})$$

where $\zeta_{t-1}^i := \lambda_{t-1}^i a_{t-1}^i - \psi_{t-1}^i$. Γ_{t-1} *is the (endogenous) joint distribution of agents across assets and* ζ_{t-1}^i*, given the optimal policies. The map* T *induces the law of motion of the distribution.*

Baseline Model: Ramsey Policy

Let $Prob(a_t^i \in \mathcal{A} \cap \zeta_t^i \in \mathcal{Z})$ denote the cumulative probability that a generic agent i has assets in the set $\mathcal{A}$ and state variable ζ_t^i in $\mathcal{Z}$. Then,

$$Prob(a_t^i \in \mathcal{A} \cap \zeta_t^i \in \mathcal{Z}) = \frac{1}{N} \sum_k \mathcal{I}(F^a \in \mathcal{A} \cap F^\lambda F^a - F^\psi \in \mathcal{Z})$$

where from Lemma 2 F^λ, F^a, F^ψ are functions of the state vector, including Γ_{t-1}. $\mathcal{I}(x)$ is an indicator variable that takes the value 1 when x is true.

The above implicitly defines the map T.

Baseline Model: Ramsey Policy

1. Usual difficulty: Γ_t is a large dimensional object. For (relatively) small and finite N one has to include the vectors $\{a_t^i\}_{i \in \mathbf{N}}, \{\zeta_t^i\}_{i \in \mathbf{N}}$ as state variables. For large or infinite N one needs an infinite number of moments.
2. Additional difficulty: Not simple to know the domain of ζ_t^i.
3. Whereas $a_t^i \in [\underline{a}, \overline{a}]$ where $\underline{a} =$ natural borrowing limit and $\overline{a} =$ buffer stock of savings? Actually, not simple to know either $+$might not exist!

Baseline Model: Analytical Results

1. Complete Markets: An example where distributions do not matter.
2. Incomplete Markets.... distributions matter.

Assumption 1: u is log and v is constant Frisch elasticity (=1). We then have:
$\widetilde{h}^i_t(\tau_t, c^i_t) := \frac{1-\tau_t}{c^i_t}\epsilon^i_t$. $\frac{d\widetilde{h}^i_t}{d\tau_t} = -u^i_{c,t}\epsilon^i_t$ and $\frac{d\widetilde{h}^i_t}{dc^i_t} = (1-\tau_t)u^i_{cc,t}\epsilon^i_t$.

Baseline Model: Analytical Results

Proposition 1 Complete Markets *Assume ϵ^i_t constant for all t. and $g_t = \overline{g}$. Consider initial conditions Γ_{-1} with $\zeta^i_{-1} = \lambda^i_{-1} = \psi^i_{-1} = 0$ and a^i_{-1} distributed according to the marginal cdf $\Gamma_{a,-1}$. Then under the assumption A1 optimal taxes satisfy: Optimal taxes satisfy:*

$$(1-\tau_0) = \frac{1}{\frac{1}{N}\sum_{i\in\mathbf{N}} \frac{h^i_0 \epsilon^i C_0}{c^i_0 H_0}\left(1+2\lambda^i\right)}\left[1+\frac{1}{N}\sum_{i\in\mathbf{N}} u^i_{c,0} a^i_{-1}\lambda^i\right]$$

$$(1-\tau_t) = 1-\overline{\tau} = \frac{1}{\frac{1}{N}\sum_{i\in\mathbf{N}} \frac{\overline{h}^i \epsilon^i \overline{C}}{\overline{c}^i \overline{H}}\left(1+2\lambda^i\right)}$$

where $\overline{X}$ ($\overline{x}^i$) is aggregate (idiosyncratic) consumption/hours in period $t = 1, 2, ...$ and $\lambda^i_t = \lambda^i$ (constant) for all t.

The optimal policy functions F^y for $t = 1, 2, ...$ can be written as $F^a(\epsilon^i, \Gamma_{-1}), F^\lambda(\epsilon^i, a^i_{-1}, \Gamma_{-1}), F^\psi(\epsilon^i, a^i_{-1}, \Gamma_{-1})$.

Baseline Model: Analytical Results

Intuitively, the budget constraints

$$a_{t-1}^i \overline{u}_c^i = \sum_{j=0}^{\infty} \beta^j \left(\overline{c}^i \overline{u}_c^i - (1-\overline{\tau}) \overline{h}^i \overline{u}^i \epsilon^i \right)$$

are slack for $t = 1, 2, ...$, Household wealth is a residual in the optimal policy program.

Same applies for ζ_t^i. From Proposition 1 $\zeta_t^i = \overline{\lambda}^i a_t^i - \psi_t^i$. We only need to show that ψ_t^i is not relevant

$$\psi_{t-1}^i = \sum_{j \geq 0} \beta^j \left(-1 + (1-\overline{\tau}) \epsilon^i \frac{d\widetilde{h}^i}{d\overline{c}^i} - \lambda^i \left((1-\overline{\tau}) \epsilon^i \widetilde{h}_t^i \overline{u}_c^i - (1-\overline{\tau}) \frac{d\widetilde{h}^i}{d\overline{c}^i} \epsilon^i \right) + \frac{\xi}{\overline{u}_c^i} (1 - \frac{d\widetilde{h}^i}{d\overline{c}^i}) \right)$$

(for a bounded solution of the Lagrange multiplier). This is also a slack variable for agent *i*s policy function.

Hence *a* and ζ are not relevant. Γ is also redundant.

Baseline Model: Analytical Results

Proposition 2 Incomplete Markets Aggregate shocks *Assume that ϵ_t^i and g_t follow random processes. Then under A1 optimal taxes satisfy: Optimal taxes satisfy:*

$$(1-\tau_t) = \frac{1}{\frac{1}{N}\sum_{i\in\mathbf{N}} \frac{h_t^i \epsilon_t^i C_t}{H_t c_t^i}\left(1+2\lambda_t^i\right)}\left[1+\frac{1}{N}\sum_{i\in\mathbf{N}} u_{c,t}^i\left[a_{t-1}^i \Delta\lambda_t^i + \psi_{t-1}^i\right]\right]$$

The optimal policy functions F^y for $t = 1, 2, ...$ are $F^a(\epsilon_t^i, g_t, a_{t-1}^i, \zeta_{t-1}^i, \Gamma_{t-1}), F^\lambda(\epsilon_t^i, g_t, a_{t-1}^i, \zeta_{t-1}^i, \Gamma_{t-1}), F^\psi(\epsilon_t^i, g_t, a_{t-1}^i, \zeta_{t-1}^i, \Gamma_{t-1})$. For $t = 0$ the policy functions are $\widetilde{F}^a(\epsilon_0^i, g_0, a_{-1}^i, 0, , \Gamma_{-1}), \widetilde{F}^\lambda(\epsilon_0^i, g_0, a_{-1}^i, 0, \Gamma_{-1}), \widetilde{F}^\psi(\epsilon_0^i, g_0, a_{-1}^i, 0, \Gamma_{-1})$.

Baseline Model: Analytical Results

Incomplete markets restores the importance of the state variables $a^i_{t-1}, \zeta^i_{t-1}, \Gamma_{t-1}$ in the policy functions.

Why?

$$\underbrace{a^i_{t-1}}_{(g^{t-1},\epsilon^i_{t-1})\text{ Measurable}} = \underbrace{\frac{1}{u^i_{c,t}}\mathbf{E}_t \sum_{j=0}^{\infty} \beta^j \left(c^i_{t+j} u^i_{c,t+j} - (1-\tau_{t+j})\epsilon^i_{t+j} h^i_{t+j} u^i_{c,t+j} \right)}_{(g^t,\epsilon^i_t)\text{ Measurable}}$$

is not a slack constraint in the planner's program.

Similarly, ψ^i_{t-1} is not a slack variable. It captures previous commitments for (state contingent) consumption in t.

Baseline Model: Analytical Results

A useful decomposition is

$$\frac{1}{N}\sum_{i\in\mathbf{N}} u^i_{c,t}\left[a^i_{t-1}\Delta\lambda^i_t + \psi^i_{t-1}\right] =$$

$$= \underbrace{\frac{1}{N}\sum_{i\in\mathbf{N}} a^i_{t-1}}_{Ea^i_{t-1}} \underbrace{\frac{1}{N}\sum_{i\in\mathbf{N}} u^i_{c,t}\Delta\lambda^i_t}_{Eu^i_{c,t}\Delta\lambda^i_t} + \underbrace{\mathrm{Cov}_i(a^i_{t-1}, u^i_{c,t}\Delta\lambda^i_t)}_{\text{Redistribution}} + \underbrace{\frac{1}{N}\sum_{i\in\mathbf{N}} u^i_{c,t}\psi^i_{t-1}}_{\text{Euler constraints}}$$

Numerical Algorithm

We solve the model by Parameterizing expectations (den Haan and Marcet (1992), Faraglia Marcet Oikonomou and Scott (2019)).

1. We setup the problem so that we can explore block linearity in idiosyncratic variables.
2. We use Condensed PEA (FMOS) to introduce a large vector of higher order moments of Γ.

Main Algorithm Given the state vector, solve FONC to find $\{c_t^i, h_t^i, \lambda_t^i, a_t^i\}_{i\in\mathbf{N}}$ and the aggregate prices τ_t, ξ_t, q_t. PEA formulates approximations

$$\mathbf{E}_t \lambda_{t+1}^i u_{c,t+1}^i \approx \Phi_\lambda\left((\epsilon_t^i, g_t, a_{t-1}^i, \zeta_{t-1}^i, \Gamma_{t-1}), \mu_\lambda\right)$$

$$\mathbf{E}_t u_{c,t+1}^i \approx \Phi_{u_c}\left((\epsilon_t^i, g_t, a_{t-1}^i, \zeta_{t-1}^i, \Gamma_{t-1}), \mu_{u_c}\right)$$

where Φ_λ and Φ_{u_c} belong to a class of functions that can approximate the conditional expectations arbitrarily well. Φ_λ and Φ_{u_c} are polynomials of a given order, μ_λ, μ_{u_c} are the coefficients.

Numerical Algorithm

Linearity in idiosyncratic variables. Solving the system of first order conditions is easy. Given the vector τ_t, ξ_t, q_t the idiosyncratic endogenous variables can be found from

$$\begin{aligned} c_t^i &= u^{-1}(\frac{\beta}{q_t}\Phi_{u_c}) \\ \lambda_t^i &= \Phi_\lambda / \Phi_{u_c} \\ a_t^i &= \frac{1}{q_t}(a_{t-1}^i - c_t^i + \epsilon_t^i \widetilde{h}_t^i (1-\tau_t)) \end{aligned} \tag{5}$$

for $i \in \mathbf{N}$. Solving the linear system (5) is particularly tractable since it is diagonal in c and λ. Then trivially substituting the vector c in the last equation we can solve for assets a. Given the solution of (5), the vector ψ can be simply obtained from

$$\begin{aligned} \psi_t^i = -\frac{1}{q_t u_{cc,t}^i}\Bigg[& u_{c,t}^i - v_{\widetilde{h}_t^i}\frac{d\widetilde{h}_t^i}{dc_t^i} - a_{t-1}^i u_{cc,t}^i \Big(\lambda_t^i - \lambda_{t-1}^i\Big) + \lambda_t^i \Big(u_{c,t}^i + c_t^i u_{cc,t}^i \\ & -(1-\tau_t)\epsilon_t^i \widetilde{h}_t^i u_{cc,t}^i - (1-\tau_t)\epsilon_t^i \frac{d\widetilde{h}_t^i}{dc_t^i} u_{c,t}^i \Big) - \xi_t (1 - \frac{d\widetilde{h}_t^i}{dc_t^i}) - u_{cc,t}^i \psi_{t-1}^i \Bigg] \end{aligned} \tag{6}$$

Numerical Algorithm

Dealing with the distribution. Two step procedure: Following KS (1998) we consider that the distribution can be accurately represented with a finite set of (non-central) moments.

$$\Upsilon_t^m = \left\{ \frac{1}{N} \sum_{i \in \mathbf{N}} \left(a_{t-1}^i \right)^r \left(\zeta_{t-1}^i \right)^k \right\}_{r,k \geq 0 \,:\, 1 \leq r+k \leq m}$$

where Υ^m denotes the set of moments of order less than equal to $m < \infty$.

Then, use Condensed PEA. First, solve the model with a core vector (means only). Second, add higher order moments as linear combinations.

Numerical Algorithm

Formally:

(Condensed PEA) Let $\Upsilon_t^{m,core} \subset \Upsilon_t^m$ denote the subset of moments that enter independently into the approximations .. Let $\Upsilon_t^{m,out} = \Upsilon_t^m \setminus \Upsilon_t^{m,core}$ denote the vector of the remaining moments.

1. Solve the model with the core state variables $X_t^{core} := \left(\epsilon_t^i, g_t, a_{t-1}^i, \zeta_{t-1}^i \Upsilon_{t-1}^{m,core}\right)$.

   Retrieve the residuals $\upsilon_{\lambda,t+1}^i$ and $\upsilon_{u_c,t+1}^i$ from the regressions
   $\lambda_{t+1}^i u_{c,t+1}^i = \Phi_\lambda(X_t^{core,i}, \mu_\lambda^{core}) + \upsilon_{\lambda,t+1}^i$ and $u_{c,t+1}^i = \Phi_{u_c}(X_t^{core,i}, \mu_{u_c}^{core}) + \upsilon_{u_c,t+1}^i$
2. Project X_t^{out} on X_t^{core} and use the residuals $\hat{\theta}_t^{out}$ to run $\hat{\upsilon}_{u_c,t}^i = \hat{\theta}_t^{out}\kappa_{u_c}$ and $\hat{\upsilon}_{\lambda,t}^i = \hat{\theta}_t^{out}\kappa_\lambda$
3. Run the model again with the linear combinations $\hat{\theta}_t^{out}\kappa_\lambda$ $\hat{\theta}_t^{out}\kappa_{u_c}$. New state vectors are $[X_t^{core}, \hat{\theta}_t^{out}\kappa_{u_c}]$ and $[X_t^{core}, \hat{\theta}_t^{out}\kappa_{u_c}]$ Iterate till convergence obtained.

Numerical Algorithm

A counterpart for the KS R^2 text

The model doesnt have a counterpart for the accuracy of the parameterized law of motion T. This is implicit in approximations of $Eu^i_{c,t+1}$ $E\lambda^i_{t+1}u^i_{c,t+1}$.

However, CPEA gives us a counterpart based on computing the gain in terms of R^2 of $Eu^i_{c,t+1}$ on X^{core}_t and on the extended vector $[X^{core}_t, \hat{\theta}^{out}_t \kappa_{u_c}]$. If $\Delta R^2 \approx 0$ higher order moments are not needed.

Numerical Solution

Baseline preferences specified in A.1. Frisch elasticity =1. Model horizon is one year. $\beta = 0.98$.

Government spending follows

$$g_t = \rho_g g_{t-1} + (1 - \rho_g)\overline{g} + \nu_{g,t}$$

where $\nu_{g,t}$ is mean zero constant variance normally distributed error term. $\overline{g}$ is 20 percent of GDP.

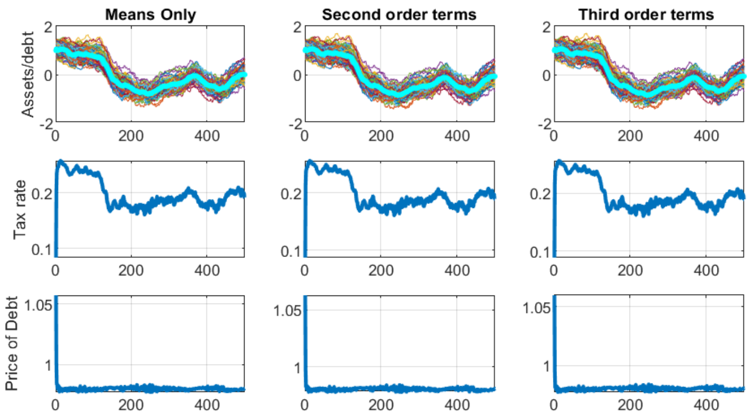
Idiosyncratic productivity follows

$$\epsilon_t^i = \rho_\epsilon \epsilon_{t-1}^i + (1 - \rho_\epsilon)1 + \nu_{\epsilon,t}$$

various calibrations. For now baseline is i.i.d shocks. So assets are very imprortant in terms of self-insurance.

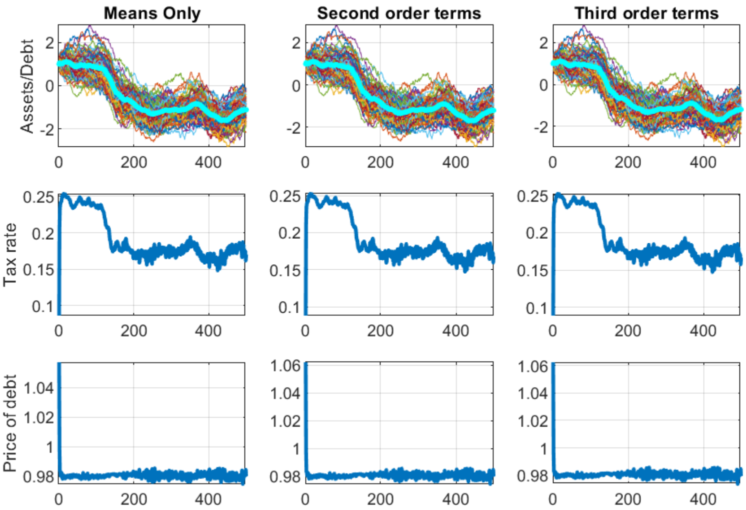
Numerical Solution

Figure: Simulation from the model 1: i.i.d Shocks



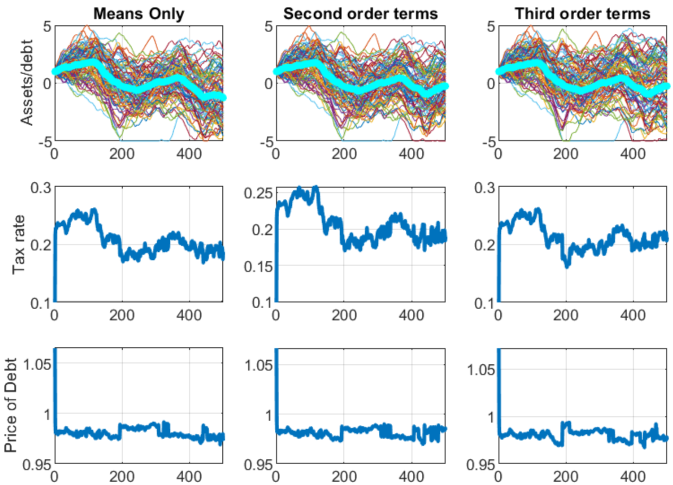
Numerical Solution

Figure: Simulation from the model 2: i.i.d Shocks



Numerical Solution

Figure: Simulation from the model 3: Persistent Shocks



Numerical Solution: Results

1. **Main result** ('Approximate Aggregation'). Only first order moments are relevant for the solution. Higher order (cross sectional) moments are unimportant.
2. As in AMSS government enganges in precautionary savings accumulation (debt becomes negative). We will see whether this holds under alternative calibrations.

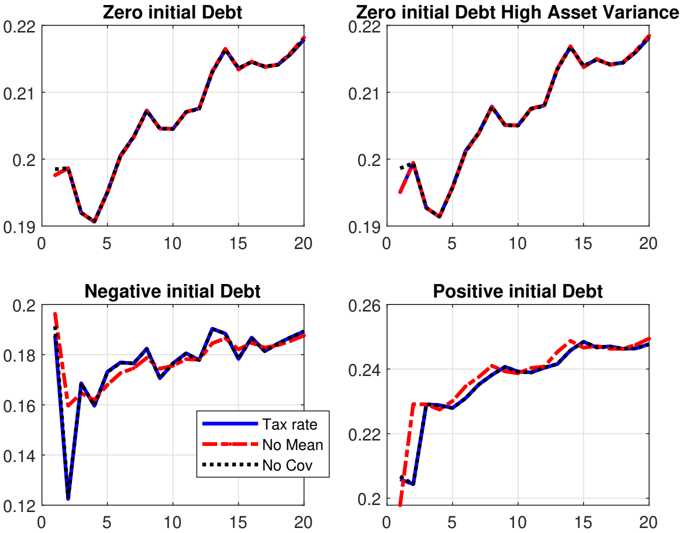
Numerical Solution

Why are only means sufficient? Lets go back to

$$\frac{1}{N}\sum_{i\in\mathbf{N}} u^i_{c,t}\left[a^i_{t-1}\Delta\lambda^i_t + \psi^i_{t-1}\right] =$$

$$= \underbrace{\frac{1}{N}\sum_{i\in\mathbf{N}} a^i_{t-1}}_{Ea^i_{t-1}} \underbrace{\frac{1}{N}\sum_{i\in\mathbf{N}} u^i_{c,t}\Delta\lambda^i_t}_{Eu^i_{c,t}\Delta\lambda^i_t} + \underbrace{\mathrm{Cov}_i(a^i_{t-1}, u^i_{c,t}\Delta\lambda^i_t)}_{\text{Redistribution}} + \underbrace{\frac{1}{N}\sum_{i\in\mathbf{N}} u^i_{c,t}\psi^i_{t-1}}_{\text{Euler constraints}}$$

Numerical Solution: Optimal Tax Decomposition



Numerical Solution: Results

1. ...
2. ...
3. Intertemporal tax distortions for the government budget constraint only, not for redistribution (except in the initial period).

Numerical Solution: Multiple Equilibria

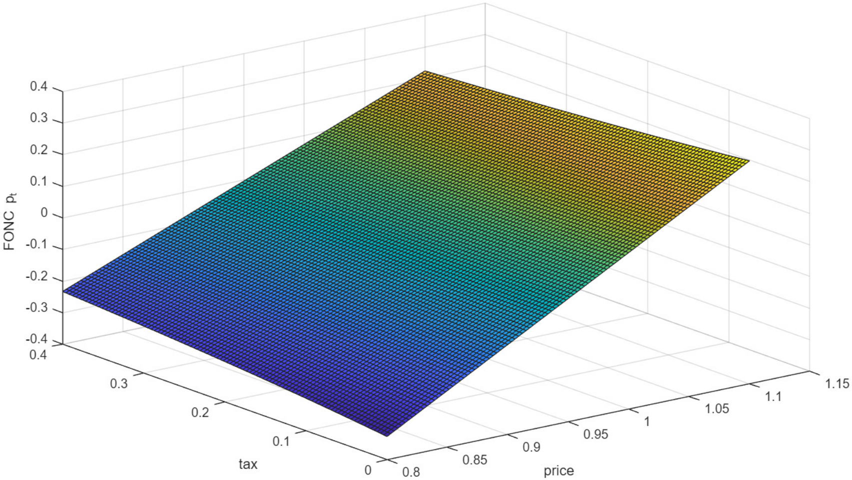
We can easily verify that there is a unique equilibrium under Ramsey policy. The system of FONC + constraints is 'static' (given the PEA approximation).

Check the resource constraint and the bond price condition for a wide range of τ_t and q_t.

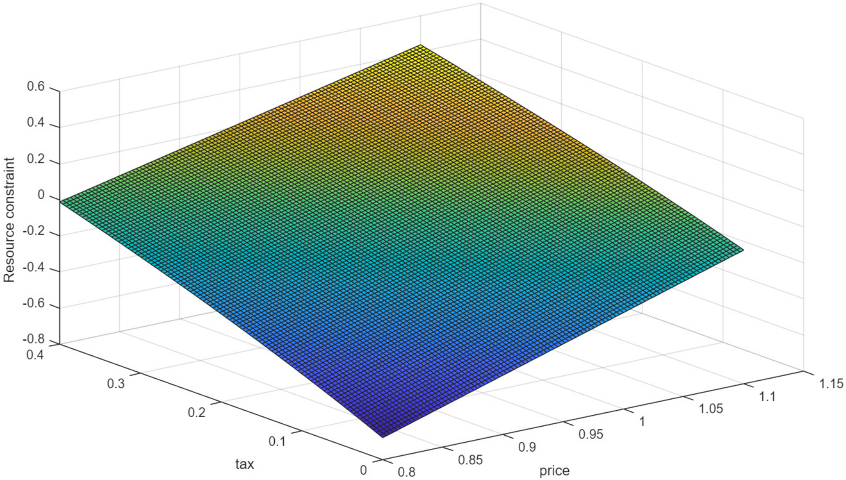
We can confirm the uniqueness of the optimum.

(This is not a fullproof argument, it hinges on the PEA approximation.)

Numerical Solution: Multiple Equilibria



Numerical Solution: Multiple Equilibria



Extension: Long term debt

If agents can trade long and short term bonds then the state vector expands considerably. The planner has to keep track of lags of distributions of assets and promises ζ. We now turn to such models.

Consider $\mathbf{J}$ is the set of assets available in the economy. We assume that $\mathbf{J} := \{1, 2, ..., J\}$.
The generic household's budget constraint is:

$$\sum_{j \in \mathbf{J}} a_t^{i,j} q_t^j = \sum_{j \in \mathbf{J}} a_{t-1}^{i,j} q_t^{j-1} + (1 - \tau_t) h_t^i - c_t^i$$

with $q_t^0 = 1$.
Moreover, we let $a_t^{i,j} \in [\underline{a}_{i,j}, \overline{a}_{i,j}]$. Debt limits can be different across agents. We want to study cases where some agents are constrained and perpahs cannot participate in a subset of markets (segmented bond markets Vayanos and company).

Extension: Long term debt

In a competitive equilibrium, and unless an $a_t^{i,j}$ is at the bound or $\underline{a}_{i,j} = \overline{a}_{i,j} = 0$ (agent i cannot purchase debt of maturity j) the price q_t^j will satisfy:

$$q_t^j u_{c,t}^i = \beta \mathbf{E}_t u_{c,t+1}^i q_{t+1}^{j-1}$$

equating the marginal cost of purchasing the asset today, $q_t^j u_{c,t}^i$ with the (expected) marginal gain of selling the asset as a $j-1$ maturity in the next period.

If a bound is hit we have

$$q_t^j u_{c,t}^i \neq \beta \mathbf{E}_t u_{c,t+1}^i q_{t+1}^{j-1}$$

Extension: Long term debt

The Lagrangian for optimal policy

$$\mathcal{L} = \mathbf{E}_0 \sum_{t \geq 0} \beta^t \Bigg\{ \sum_{i \in \mathbf{N}} \left(u(c_t^i) - v(\widetilde{h}_t^i) \right) + \lambda_t^i \Bigg(\sum_j q_t^j a_t^{i,j} u_{c,t}^i - \sum_j q_t^{j-1} a_{t-1}^{i,j} u_{c,t}^i + c_t^i u_{c,t}^i$$

$$-(1-\tau_t)\epsilon_t^i \widetilde{h}_t^i u_{c,t}^i \Bigg) - \xi_t \left(\sum_{i \in \mathbf{N}} c_t^i + g_t - \sum_{i \in \mathbf{N}} \widetilde{h}_t^i \epsilon_t^i \right) + \sum_{j \in \mathbf{J}} \widetilde{\psi}_t^{i,j} \left(q_t^j u_{c,t}^i - \beta \mathbf{E}_t u_{c,t+1}^i q_{t+1}^{j-1} + \nu_t^{i,j} \right)$$

$$+ \sum_{j \in \mathbf{J}} \underline{\omega}_t^{i,j} (a_t^{i,j} - \underline{a}_{i,j}) + \sum_{j \in \mathbf{J}} \overline{\omega}_t^{i,j} (\overline{a}_{i,j} - a_t^{i,j}) \Bigg\} + CS$$

where $\nu_t^{i,j}$ is (the sum of) the Lagrange multipliers on inequality constraints. Terms CS take care of the complementary slackness conditions (omitted for brevity).

Extension: Long term debt

The first order conditions characterizing the optimum are given by:

$$u^i_{c,t} - v_{\widetilde{h}^i_t}\frac{d\widetilde{h}^i_t}{dc^i_t} - \xi_t(1 - \frac{d\widetilde{h}^i_t}{dc^i_t}\epsilon^i_t) + \lambda^i_t\left(u^i_{c,t} + c^i_t u^i_{cc,t} - (1-\tau_t)\epsilon^i_t\widetilde{h}^i_t u^i_{cc,t} - (1-\tau_t)\epsilon^i_t\frac{d\widetilde{h}^i_t}{dc^i_t}u^i_{c,t}\right)$$

$$+u^i_{cc,t}\lambda^i_t\left(\sum_j q^j_t a^{i,j}_t - \sum_j q^{j-1}_t a^{i,j}_{t-1}\right) + u^i_{cc,t}\sum_j\left(\widetilde{\psi}^{i,j}_t q^j_t - \widetilde{\psi}^{i,j}_{t-1}q^{j-1}_t\right) = 0 \quad \text{for} \quad i \in \mathbf{N}$$

$$q^j_t\lambda^i_t u^i_{c,t} = E_t\beta q^{j-1}_{t+1}u^i_{c,t+1}\lambda^i_{t+1} + \underline{\omega}_{i,j} - \overline{\omega}_{i,j} = 0$$

$$\sum_{i\in N} u^i_{c,t}\left(a^{i,j}_t\lambda^i_t - a^{i,j+1}_{t-1}\lambda^i_t + \widetilde{\psi}^{i,j}_t - \widetilde{\psi}^{i,j+1}_{t-1}\right) = 0 \quad \text{for} \quad j < J$$

$$\sum_{i\in N} u^i_{c,t}\left(a^{i,J}_t\lambda^i_t + \widetilde{\psi}^{i,J}_t\right) = 0 \qquad \widetilde{\psi}^{i,j}_t = 0 \quad \text{if} \quad a^{i,j}_t \in \{\underline{a}_{i,j}, \overline{a}_{i,j}\}$$

In terms of the initial multipliers it holds that: $\widetilde{\psi}^{i,j}_t + a^{i,J}_t\lambda^i_t = \psi^{i,j}_t$.

Extension: Long term debt

The state vector is

$$\left(\epsilon^{i}_{t,t}\,, \left\{a^{i,j}_{t-k}\right\}_{j=1:J,\; j\geq k\geq 0}, \left\{\zeta^{i,j}_{t-k}\right\}_{j=1:J,\; j\geq k\geq 0}, \left\{\Gamma_{t-k}\right\}_{k\leq J}\right)$$

Hence many lags of the distributions are state variables.

The CPEA can handle the expanded state vector!

Extension: Long term debt

When will distributions matter most?

Proposition 3. Consider an economy with J assets, loose debt limits for all i, ϵ_t^i constant and the cardinality of G is $\leq J$. Assume further that the matrix

$$\widetilde{Q}_{\mathbf{J},t} = \begin{bmatrix} 1 & \beta \mathbf{E}_t\left(u^i_{c,t+1}|s_t = s_1\right) & \beta^2 \mathbf{E}_t\left(u^i_{c,t+2}|s_t = s_1\right) & \ldots & \beta^J \mathbf{E}_t\left(u^i_{c,t+J}|s_t = s_1\right) \\ \vdots & \ldots & \ddots & \ldots & \vdots \\ 1 & \beta \mathbf{E}_t\left(u^i_{c,t+1}|s_t = s_{n_G}\right) & \beta^2 \mathbf{E}_t\left(u^i_{c,t+2}|s_t = s_{n_G}\right) & \ldots & \beta^2 J \mathbf{E}_t\left(u^i_{c,t+J}|s_t = s_{n_G}\right) \end{bmatrix} \tag{7}$$

is of full rank. Then, for any $t > J - 1$, the distributions Γ do not matter for allocations.

Extension: Long term debt

Proposition 3 is basically a complete market outcome. When there are enough maturities to span g the planner can complete the market. (Such an allocation can be characterized analytically; online appendix.)

The corollary is that we need to constrain trades in assets to make distributions matter. Let us focus on lower debt limits (short sale constraints) since this makes most sense. The next result show that this might not work either...

Extension: Long term debt

Proposition 4. Consider the case where $|G| = 2$ (cardinality is 2) and let $g_t \in \{g_L, g_H\}$. Assume that $J = 2$. Moreover, consider the case where the lower debt limits are tight (for example) $\underline{a}_{i,j} = 0$. Then, almost surely the economy converges to a complete market, $\lambda_t^i \to \overline{\lambda}^i$ a.s.

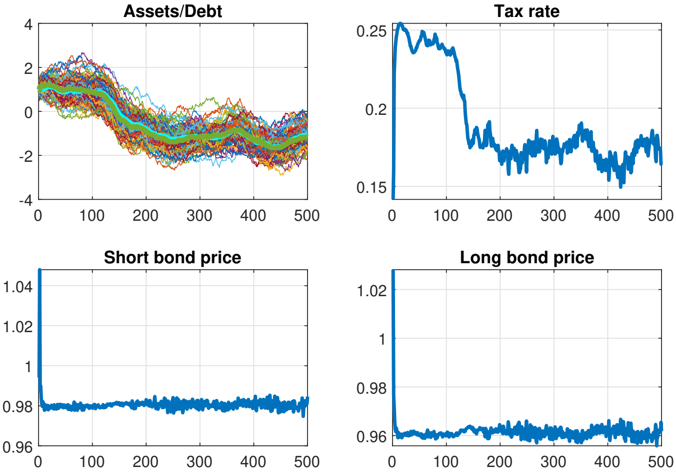
Extension: Long term debt

We consider a model with fully segmented bond markets. $\mathbf{N} = \bigcup_{j=1}^{\mathbf{J}} \mathbf{N}_j$ where $\mathbf{N}_j$ are disjoint sets.

So agents in $\mathbf{N}_j$ can only trade in bonds of maturity j.

Extension: Long term debt

Figure: Simulation from the model 4: iid Shocks



Conclusions

Thanks!