Learning Networks

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Peers Influenced Behaviors

- Behaviors are influenced by, or chosen to coordinate, with peers.
  - Languages to learn,
  - Technology adoption,
  - Effort into community activity.
- Not only direct influence but also indirect influence is important:
  - Those influencing me are themselves influenced by others
- Rational decision making hinges on full network architecture

Challenge of Incomplete Information

- Access to complete information about the network structure is rare
- Agents know their direct neighbors but not the entire network
  - Firms may not know their suppliers' suppliers
  - Social media users don't know their friends' friends
- Even so, behavior under incomplete information encodes adjacency structure (CJST (2024))
- If agents behavior encodes how they are connected, and agents know this, can a network be learned simply by observing others behavior?

This Paper

- Establish a simple theoretical benchmark for learning networks.
- Given an infinitely repeated network game which parametrizes behavior, under what conditions can a network be learned?

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- Given an infinitely repeated network game which parametrizes behavior, under what conditions can a network be learned?
- **Main Result:** Network learnability is associated with injectiveness of network statistics which parametrize agent behavior in-game

Related Literature

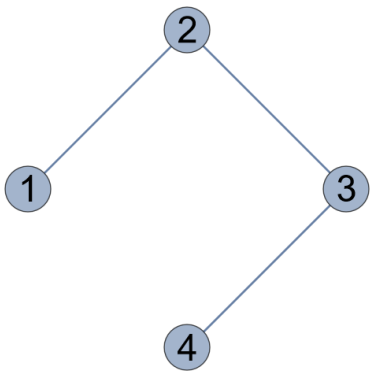
- Network Games Under Incomplete Information:
  - Chaudhuri, Jackson, Sarangi, and Tzavellas (2024)
- Bayesian Learning in Games:
  - Kalai and Lehrer (1993), and Mueller-Frank (2013)

Model Overview

- Agents are myopic but rational
- Nature chooses a network from a known distribution
- Each player sees who their direct neighbors are in the network
- Players get to observe the actions of their neighbors
- After observing actions, players update their beliefs about the network and choose their next actions accordingly
    - Last point repeats infinitely

Time and Ex-Ante Beliefs

- $N = \{1, 2, \ldots . n\}$ player set, $t \in \{1, 2, \ldots\}$ time
- $\mathcal{G}$: A finite set of possible networks on N
- Nature chooses $g \in \mathcal{G}$ according to some $p \in \Delta(\mathcal{G})$
- $\mathcal{G}$ and p are common knowledge
- Example, $p = (p(g_a), p(g_b), p(g_c), p(g_d)) = (\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$ with



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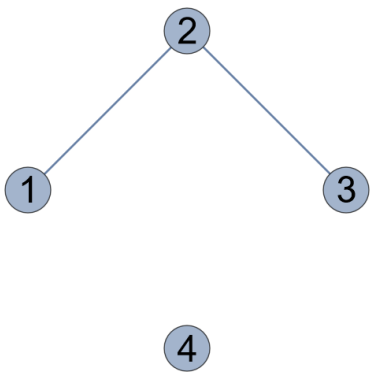

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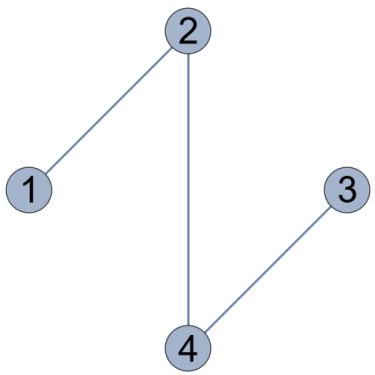

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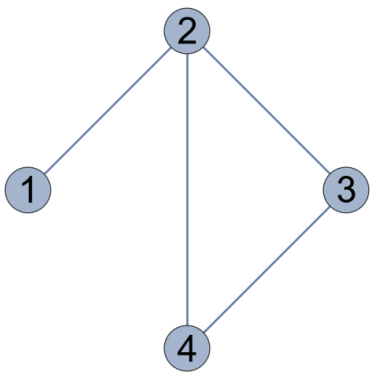

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Representing Network Uncertainty

- At each t each agent i has a partition of $\mathcal{G}$ denoted by

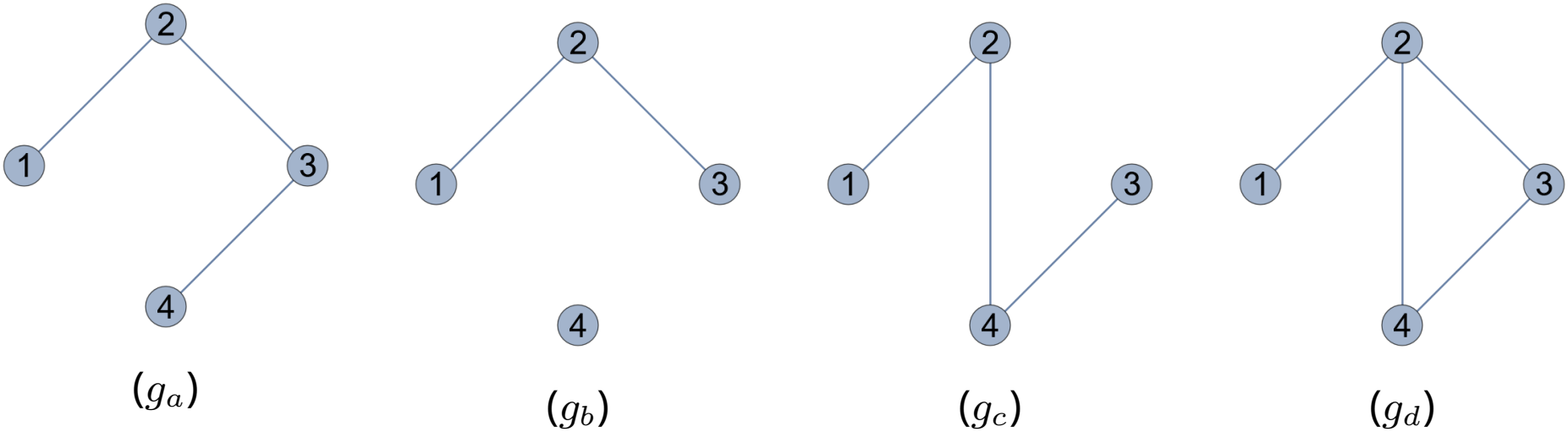
$$T_i^t = \{T_{i(g)}^t | g \in \mathcal{G}\}$$

- $T_{i(g)}^t$ is the cell of $i's$ partition containing g in period t.
- T_i^t is the ***type set*** of player i at time t and $T^t = \times_{i \in N} T_i^t$ the ***type space***
- Denote by $I_i^t \in T_i^t$ is the ***realized type*** of player i at time t

Definition

We interpret $T_{i(g)}^t$ as ***the set of all networks that are indistinguishable from*** g for player i in period t conditional on all its available information.

Neighborhood Induced Partition



- In period 1, only available info comes from neighborhoods:

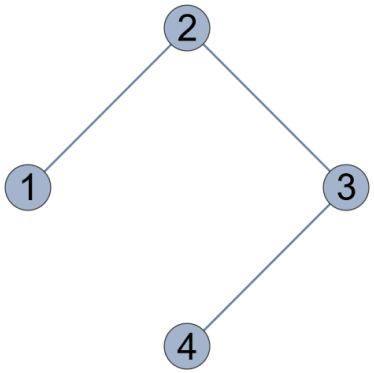
$$T_i^1 = \{g' \in \mathcal{G} | g'_{ji} = g_{ji}, \forall j \in N\}$$

- $N_2(g_a) = N_2(g_b) = \{1, 3\}$, $N_2(g_c) = \{1, 4\}$, and $N_2(g_d) = \{1, 3, 4\}$
- $T^1_{2(g_a)} = T^1_{2(g_b)} = \{g_a, g_b\}$, $T^1_{2(g_c)} = \{g_c\}$, and $T^1_{2(g_d)} = \{g_d\}$
- When g_a is realized, $I_2^1 = \{g_a, g_b\}$
- $T^1 = \times_{i \in N} T_i^1$ common knowledge, but I_i^1 private information

Assigning Probabilities to Networks

- For player i with realized type $I_i^t = T_{i(g)}^t$:

$$p(g'|I_i^t = T_{i(g)}^t) = \frac{p(g')}{\sum_{g'' \in T_{i(g)}^t} p(g'')}$$



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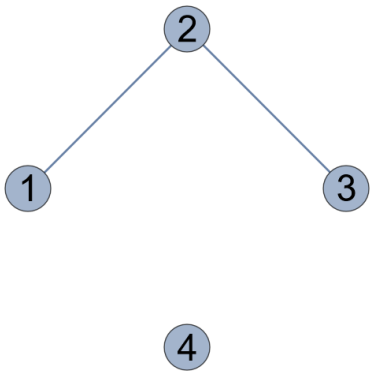

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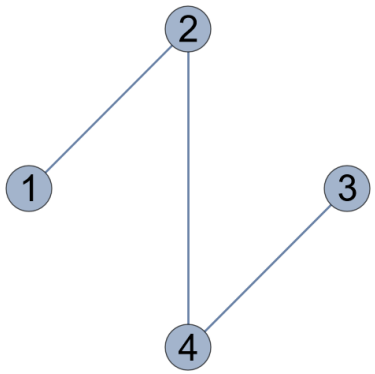

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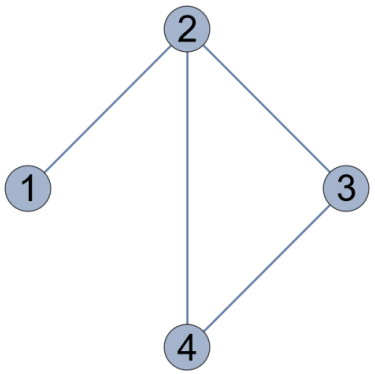

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- Suppose $p = (p(g_a), p(g_b), p(g_c), p(g_d)) = (\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$
- If g_a is the true graph, then $I_2^1 = \{g_a, g_b\}$
- So $p(g_a|I_2^1 = \{g_a, g_b\}) = \frac{1}{3}$ and $p(g_b|I_2^1 = \{g_a, g_b\}) = \frac{2}{3}$

Stage Game Payoffs

- In each stage players simultaneously exert $a_i \in A$
- Player i's per period payoff is:

$$u_i(a_i, (a_j)_{j \in N_i(g)}, g)$$

- Assume local spillovers

$$u_i(a_i, (a_j)_{j \in N_i(g)}, g) = u_i(a_i, (a_j)_{j \in N_i(g')}, g') \iff g_{ij} = g'_{ij} \; \forall i \in N$$

- For example, consider the following utility function (Ballester et al. 2006)

$$u_i(a_i, (a_j)_{j \in N_i(g)}, g) = a_i - \frac{1}{2}a_i^2 + \lambda a_i \sum_{j=1}^{n} g_{ij} a_j$$

Expected Payoff and Equilibrium Actions

- Since $u_i(a_i, (a_j)_{j \in N_i(g)}, g)$, then Harsanyi trasnformation yields:

$$E[u_i | I_i^t = T_{i(g)}^t] = \sum_{g' \in \mathcal{G}} p(g' | I_i^t = T_{i(g)}^t) u_i(a_i, (a_j(I_j^t = T_{j(g')}^t))_{j \in N_i(g')}, g')$$

Definition

A strategy for player i in period t is a map

$$\sigma_i^t : T_i^t \to A^{T_i^t}.$$

A strategy profile $\sigma^{*t} = (\sigma_i^{*t}, \sigma_{-i}^{*t})$ is a BNE of the t^{th} stage game if:

$$a_i^{*t}(I_i^t = T_{i(g)}^t) \in \underset{a_i^t}{\operatorname{argmax}} E[u_i | I_i^t = T_{i(g)}^t], \forall i \in N, \ \forall T_{i(g)}^t \in T_i^t$$

- Assume unique BNE for every stage of the game
- We focus on learning dynamics induced by $\{\sigma^{*1}, \sigma^{*2}, \dots\}$

Type Updating and Rationalizability

- Connected players observe each other's actions
- Actions are type-dependent
- Players can extract information from neighbors' actions

Definition

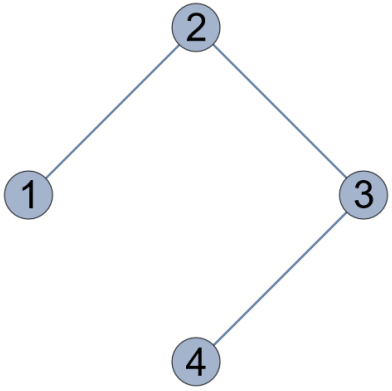
The set of types that rationalize player j's actions:

$$B^t(a_j^t) = \{T_{j(g')}^t \in T_j^t | a_j^t(I_j^t = T_{j(g')}^t) \in \underset{a_j^t \in A}{\operatorname{argmax}} E[u_j | I_j^t = T_{j(g)}^t]\}$$

Moreover, let

$$B_f^t(a_j^t) = \bigcup_{T_{j(g')}^t \in B^t(a_j^t)} T_{j(g')}^t$$

- B^t gives all possible types that induce the observed action
- B_f^t gives all the networks in B^t



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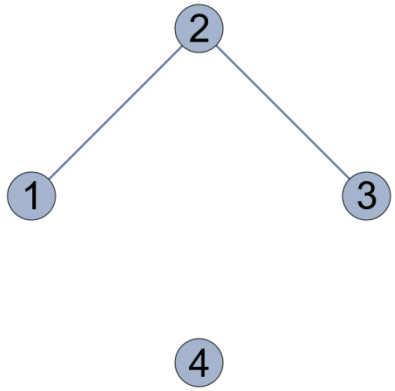

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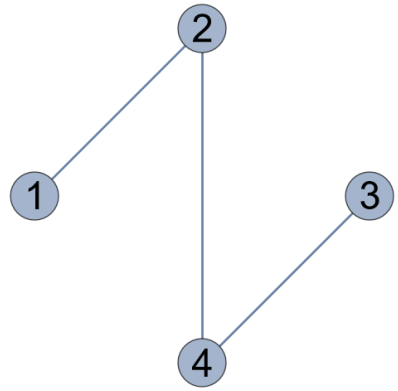

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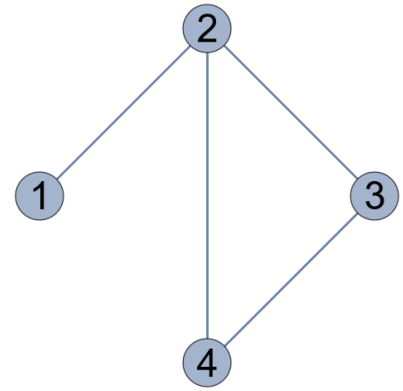

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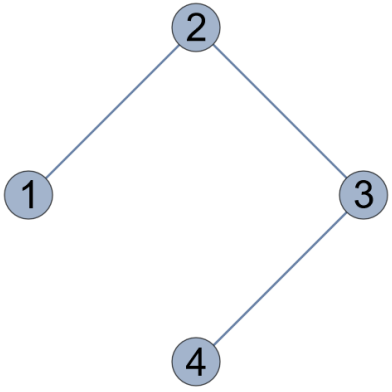
- Consider a network game under the following equilibrium values:

$$a_2^1(I_2^1 = T^1_{2(g_a)}) = a_2^1(I_2^1 = T^1_{2(g_b)}) = 1.789$$
$$a_2^1(I_2^1 = T^1_{2(g_c)}) = 1.877$$
$$a_2^1(I_2^1 = T^1_{2(g_d)}) = 2.387$$

- T_i^1 are common knowledge, so these values can be computed by all
- Under g_a realization, players 1 and 3 are connected
- Player 1 observes that 2 exerts $a_2^1 = 1.789$



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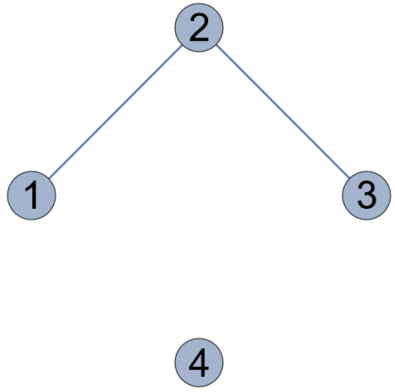

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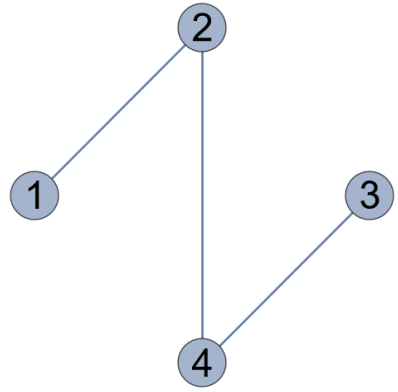

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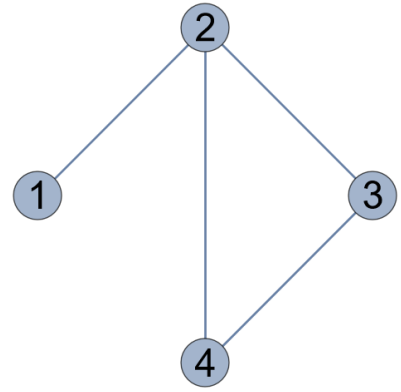

(g_d)

- The set of types that rationalize player 2's action is

$$B^1(a_2^1) = \{T^1_{2(g_a)}, T^1_{2(g_b)}\} = \{\{g_a, g_b\}\}$$

- Since player 2 is only neighbor of player 1,

$$B^1_f = \{g_a, g_b\}$$

- In period 2, player 1 updates its belief according to:

$$I^2_1 = I^1_1 \cap B^1_f = \{g_a, g_d, g_c, g_d\} \cap \{g_a, g_b\} = \{g_a, g_b\}$$

Type Updating Rule

Definition

Player i updates its realized type:

$$\underbrace{I_i^t}_{\text{Updated}} = \underbrace{I_i^{t-1}}_{\text{orginal}} \cap \underbrace{\left(\bigcap_{j \in N_i(g^{I_i^{t-1}})} (B_f^{t-1}(a_j^{t-1}(I_j^{t-1} = T_{j(g^*)}^{t-1}))) \right)}_{\text{New information from neigbhors}}$$

Updating rule for $T_{i(g)}^t$:

$$T_{i(g)}^t = \begin{cases} \{g' \in \mathcal{G} | g'_{ji} = g_{ji}, \forall j \in N\} & t = 1 \\ T_{i(g)}^{t-1} \cap \left(\bigcap_{j \in N_i(g^{T_{i(g)}^{t-1}})} (B_f^{t-1}(a_j^{t-1}(I_j^{t-1} = T_{j(g)}^{t-1}))) \right) & t = 2, 3, 4, \ldots \end{cases}$$

- Recall $T^1 = \times_{i \in N} T_i^1$ common knowledge $\implies$ σ^{*1} common knowledge
- σ^{*1} common knowledge $\implies$ $T^2 = \times_{i \in N} T_i^2$ common knowledge and so on
- Hence, we have common knowledge of type filtration

End of Learning Process

Definition

Learning ends if there exists a t^* such that $\forall t \geq t^*$:

$$T^t_{i(g)} = T^{t^*}_{i(g)}, \quad \forall i \in N, \forall g \in \mathcal{G}$$

- Includes players' realized types $I^t_i = T^t_{i(g^*)}$

Lemma

There exists a finite t^* such that learning ends

- Follows from finiteness of $\mathcal{G}$ and information filtration

Equilibrium Actions After Learning Ends

Lemma

Suppose that learning ends at t^*. For any $t \geq t^*$, and for any $g', g'' \in T^t_{i(g)}$, $a^t_j(I^t_j = T^t_{j(g')}) = a^t_j(I^t_j = T^t_{j(g'')})$, $\forall\, i \in N$, $\forall\, j \in N_i(g)$, and $\forall\, g \in \mathcal{G}$

- If $g', g'' \in T^t_{i(g)}$, both $T^t_{j(g')}$ and $T^t_{j(g'')}$ are possible types for player j
- If $a(I^t_j = T^t_{j(g')}) \neq a(I^t_j = T^t_{j(g'')})$, then player i could still learn
- Implication:

$$E[u_i | I^t_i = T^t_{i(g)}] = \sum_{g' \in \mathcal{G}} p(g' | I^t_i = T^t_{i(g)}) u_i(a_i, (a_j(I^t_j = T^t_{j(g')}))_{j \in N_i(g')}, g')$$

$$= u_i(a_i, (a_j(I^t_j = T^t_{j(g)}))_{j \in N_i(g)}, g)$$

Convergence to Complete Information Equilibrium

Proposition

Suppose that Nature has selected g and that learning ends at t^*. Then, $a_i^{t^*}(I_i^{t^*} = T_{i(g)}^{t^*}) = a_i^c(g), \forall i \in N$, where $a_i^c(g)$ is the NE of player i in the complete info version of the game under the graph g.

- Players will act as if they have complete information
- All networks in converged info set lead to the same action
- When is this set a singleton?

Learning the Network Perfectly

Definition

Learning is perfect for player i if $I_i^{t^*} = \{g^*\}$.

- That is player i learns the network.

Theorem

Let $a_i^c(g)$ be an NE of a complete information game under $g \in \mathcal{G}$. If

- $a_i^c(g) \neq a_i^c(g')$ *for all* $g, g' \in I_i^1$ *or*
- $\exists\, j \in N_i(g)$ *such that* $a_j^c(g) \neq a_j^c(g')$ *for all* $g, g' \in I_i^1$,

then player i can achieve perfect learning

- Network game equilibria are network statistics $a_i^c : \mathcal{G} \to \mathbb{R}$.
- Learnability associated with uniqueness of network statics

Also In the Paper

- Characterize learning time
- Illustration in the classical LQ game
- Compare convergence time with that of behavioral rules
- Learning with homophily
- Study key learners

Illustration in Linear-Quadratic Game

- Consider the linear-quadratic game (Ballester et al. (2006)):

$$u_i = a_i - \frac{1}{2}a_i^2 + \lambda \sum_{j \in N} g_{ij} a_j a_i$$

- NE action of player i is $a_i^c(g) = b_i(g)$, where:

$$b_i(g) = \sum_{k=0}^{\infty} \lambda^k [g^k 1]_i, \text{ and } [g^k 1]_i \text{ is all length } k \text{ walks from player } i$$

Proposition

Suppose $g \neq g'$. If (i) $0 < \lambda < 1/\max\{\rho(\mathbf{g}), \rho(\mathbf{g}')\}$, and (ii) $\exists\, k \geq 1$ such that $[g^k 1]_i \neq [g'^k 1]_i$, then $b_i(g) \neq b_i(g')$ for all λ up to a set of measure zero

- With linear-quadratic utility we have (generic) perfect societal learning

Learning time

Proposition

Learning ends in stage $t^* \leq d(\mathcal{G}) + 1$, where $d(\mathcal{G}) = \max_{g \in \mathcal{G}} d(g)$ and $d(g)$ is the diameter of the graph g.

- At stage 2, a player uses neighbors' information
- At stage 3, a player (implicitly) uses neighbors' neighbors' information
- At $(d(g) + 1)$ stage, all possible information has flowed to i

Lemma

Convergence is $\mathcal{O}(n)$.